IB Subject Area: Mathematics

Buffon's Needle: Notable extensions and their significance in the mathematical world

Research Question: What are some extensions of Buffon's Needle and what significance do they carry in the mathematical world?

October 26, 2020

Word Count: <4000

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Introduction

Buffon's Needle was a problem first posed in 1733 by mathematician Georges-Louis Leclerc, Comte de Buffon, who later recreated it with a solution in 1777.[1] At its most basic level, the experiment/problem first establishes an infinite plane with parallel lines separated by distance *d*. Needles of length *l* are then dropped or placed randomly on the plane. The problem then asks what the probability that a needle lands crossing a line.[2]

While this problem seemingly has no relationship to circles, π surfaces in this problem. When the distance *d* between parallel lines is equal to the length *l* of the needle, the probability is exactly $\frac{2}{\pi}$.[3] In this investigation, I will address how and why π comes up by proving the result of the experiment, discuss how Buffon's Needle set a precedent for the future of mathematical experiments, and address how this method was utilized to calculate π before modern computers were available. Additionally, I will discuss and find solutions to a variety of notable extensions of Buffon's needle: cases where the length of the needle is less than or greater than the distance between lines, cases where circular coins are dropped, and cases where coins and needles are dropped onto planes of repeating polygons instead of parallel lines.

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[1] Eric W. Weisstein, "Buffon's Needle Problem," Wolfram MathWorld, accessed May 1, 2020, https://mathworld.wolfram.com/BuffonsNeedleProblem.html.
[2] Lee Badger, "Lazzarini's Lucky Approximation of Pi," *Mathematics Magazine* 67, no. 2 (April 1994): 83, https://doi.org/10.2307/2690682.
[3] Weisstein, "Buffon's Needle," Wolfram MathWorld.

Buffon's Needle

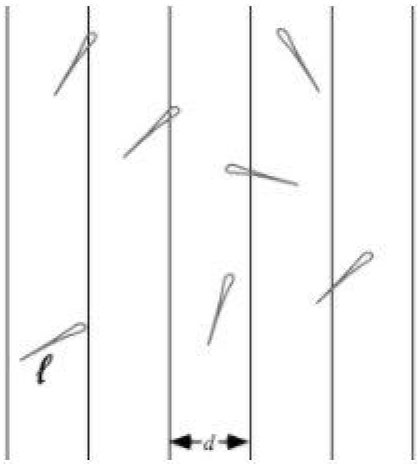

Figure 1

Figure 1 illustrates how the experiment is set up.[4] The vertical lines are separated by distance *d* and the needles are all of uniform length *l*. The needles shown are randomly dropped on the plane. We will first be analyzing the most simple case, in which $l = d = 1$. In this situation, there are 2 variables in the experiment: the distance between the center of the needle to the nearest line (denoted as *c*) and the angle θ from the center of the needle from the parallel lines. The needle shown does not cross a line.

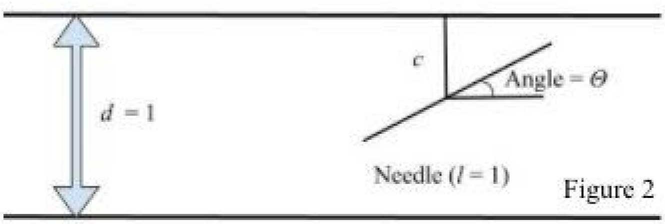

Figure 2

However, it will cross the line if $c \leq \frac{1}{2}\sin(\theta)$, as seen in the diagram to the left. The question then becomes: how often will $c \leq \frac{1}{2}\sin(\theta)$?

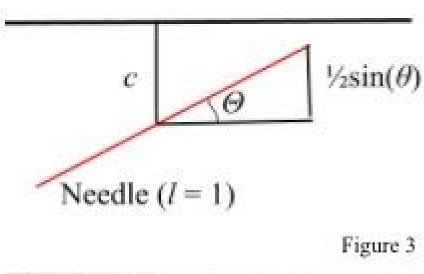

Figure 3

To answer this question, we can graph the inequality as a function $f(x) = \frac{1}{2}\sin(\theta) \leq c$, where c is ½, because $d = 1$.[5] The values on or below the curve satisfy the inequality. To solve the inequality we can solve for the area under the function using an integral as follows:

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[4] "Buffon's Needle," chart, Math Images, March 12, 2012, accessed September 28, 2020, https://mathimages.swarthmore.edu/index.php/Buffon%27s_Needle.
[5] George Reese and Pavel Safronov, Area under f(x) = 1/2sin(x), chart, MSTE, accessed September 14, 2020, https://mste.illinois.edu/activity/buffon/.

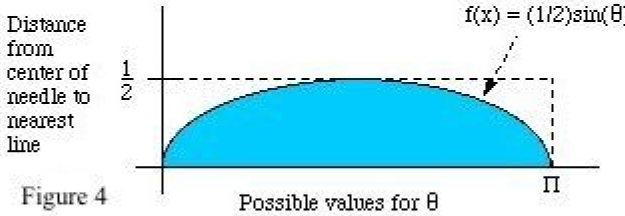


Figure 4

$$\int_0^{\pi} \frac{sin(\theta)}{2} d\theta = ½ \int_0^{\pi} sin(\theta) d\theta = ½ \left(-cos(\theta)\right)\Big|_0^{\pi}$$

$$= ½\left[-cos(\pi) - (-cos(0))\right] = ½\left[1 - (-1)\right] = ½(2) = 1$$

Now that we have found the area under the curve to be exactly equal to 1, we can divide it by the area of the rectangle to find the probability of the needle crossing a line. The area of the rectangle is simply length*width = ½* π = π/2. The final probability is 1/π/2 = 2/π ≈ .6366 ≈ 63.66%. Thus, π can be calculated by multiplying the total number of needle drops by 2 and then dividing by the number of times a needle crossed a line. The ratio 2/π is known as Buffon's Constant and is prominent in several mathematical areas such as infinite sums/products and randomly generated polynomials.[6]

[6] Neil Sloane, "Buffon's Constant," *OEIS*, last modified June 6, 2012, https://oeis.org/wiki/Buffon%27s_constant.

Extensions in Variation of Needle Length

Now that we have solved the problem when the length of the needle is equal to the distance between the lines, the problem can be expanded when we ask what the solution is when the needle is both shorter and longer than the distance between the lines.

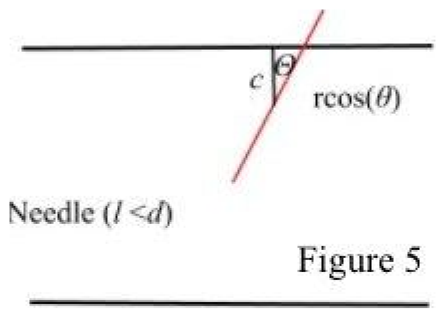


Figure 5

We will start with the scenario in which the length of the needle is shorter than the distance between the lines. First, define the ratio r as l/d. The probability can be computed with a similar integral as in the previous problem.[7] However, we will utilize a cosine function because the triangle in question is more easily recognized when flipped to the opposite side. The integral is the result of our original multiplied by the chance that the needle is long enough from the center points. Once again, the possible θ angles range from 0 to $\pi/2$ because of symmetry. The computation is as follows.

$$2/\pi \int_0^{\pi/2} r\cos(\theta)d\theta = \frac{2l}{\pi d}\int_0^{\pi/2} \cos(\theta)d\theta = \frac{2l}{\pi d}\sin(\theta) \text{ from 0 to } \pi/2 = \frac{2l}{\pi d}(\sin(\pi/2) - (\sin(0))$$

$= \frac{2l}{\pi d} = 2r/\pi$ Thus, the probability of a needle of length $l<d$ crossing a line is $\frac{2r}{\pi} = \frac{2l}{\pi d}$

[7] J. V. Uspensky, *Introduction to Mathematical Probability* (n.p.: Digital Library Of India, 1937), 252, https://doi.org/2015.263184.

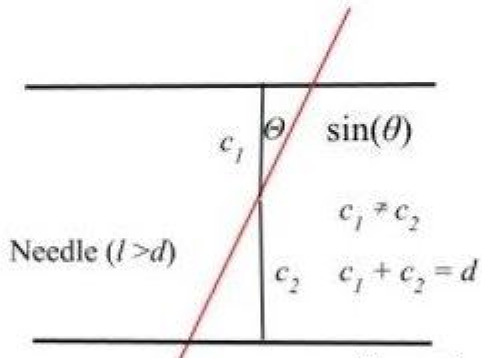


Figure 6

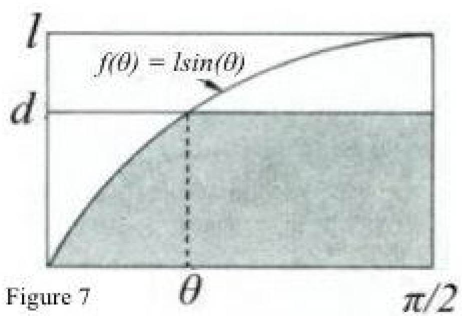


Figure 7

The scenario where the length of the needle is longer is considerably more involved. This is because the needle has a chance to cross two lines while the question simply tests if it crosses any line at all. The probability of crossing 2 lines must then be converted to that of the probability of crossing any line at all. We once again define $r = l/d$.

Figure 7 portrays a visual representation of this probability where the probability is the area of the shaded region divided by the smaller rectangle.[8]

Note that $\theta = sin^{-1}(d/l)$

Therefore the total probability can be computed as follows:

$$p = \left(l \int_0^{arcsin(d/l)} sin(\theta)d\theta + d\left[\tfrac{\pi}{2} - arcsin(d/l)\right]\right) / \tfrac{d\pi}{2}$$

$$= \tfrac{2l}{d\pi}\left[-cos(\theta)\Big|_0^{arcsin(d/l)} + d\left[\tfrac{\pi}{2} - arcsin(d/l)\right]/l\right]$$

$$= \tfrac{2l}{d\pi}[-cos(arcsin(d/l)) - (-cos(0))] + \tfrac{2}{d\pi}k(\tfrac{\pi}{2} - arcsin(d/l))$$

$$= \tfrac{2l}{d\pi}[-cos(arcsin(d/l))] + \tfrac{2l}{d\pi} + 1 - \tfrac{2}{\pi}(arcsin(d/l))$$

Now, $cos(arcsin(\tfrac{d}{l})) = cos(\theta)$, where $\theta = arcsin(d/l)$.

By using the identity $cos^2(\theta) + sin^2(\theta) = 1$, we have $k^2/d^2 + cos^2(\theta) = 1$

Thus $cos^2(\theta) = 1 - k^2/d^2$

[8] Lee L. Schroeder, "Buffon's Needle Problem: An Exciting Application of Many Mathematical Concepts," *The Mathematics Teacher*, 2nd ser., 67 (February 1974): 183-185, https://www.jstor.org/stable/27959621.

$$\text{Therefore, } cos(\theta) = \sqrt{1 - k^2/d^2} = \sqrt{\frac{l^2 - d^2}{l^2}} = \sqrt{l^2 - d^2}/l$$

$$\text{Hence, } cos(arcsin(\tfrac{d}{l})) = \sqrt{l^2 - d^2}/l$$

We can substitute this equation into our original computation for

$$p = \frac{2l}{d\pi}\left[-\sqrt{l^2 - d^2}/l\right] + \frac{2l}{d\pi} + 1 - \frac{2}{\pi}(arcsin(d/l))$$

$$= [\tfrac{2l}{d\pi} + 1] - \tfrac{2}{\pi}(\sqrt{l^2 - d^2}/l + arcsin(d/l))$$

Compared to the formula for scenarios where $l \leq d$, this is a far more complex solution, but far more intriguing! The basic answer for all 3 is the same. All are some function of Buffon's Constant, $2/\pi$.

Buffon's Needle on Square Tiles

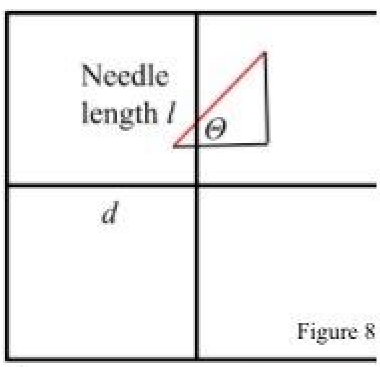


Figure 8

The first extension we will look at is one where the needle is dropped onto square tiles with side length *d*. For this case, we will assume that $l \leq d$, and that $0 \leq \theta \leq \pi/2$, (once again by symmetry all other angles can be represented within the bounds). The probability is the same as the first case $lsin(\theta)$ with the added chance that it crosses a line that is perpendicular to the first parallel set $lcos(\theta)$. Figure 9 graphically represents the probability: the area of the shaded region divided by the entire rectangular region.[9] The graph demonstrates that the

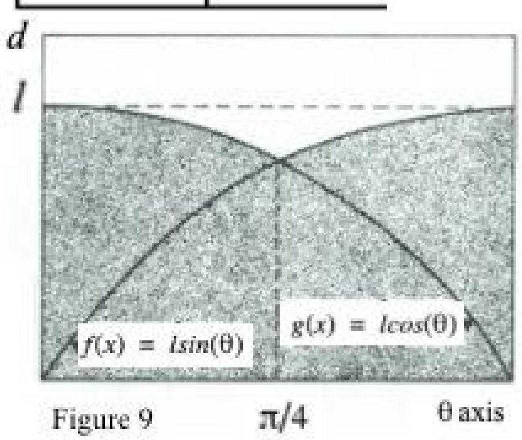


Figure 9

[9] Schroeder, "Buffon's Needle," 185-186.

chance that a needle crosses a line is far greater than the previous cases we have analyzed. We employ both the cosine and the sine function because the needle has essentially a 2nd chance at crossing a line because the set of potential lines has been "flipped" over an imaginary line that is at angle 45° to all lines. This makes the solution slightly more involved.

$$p = \left[\int_0^{\pi/4} l\cos(\theta)d\theta \ + \int_{\pi/4}^{\pi/2} l\sin(\theta)d\theta \right] / \left(\frac{d\pi}{2}\right)$$

$$= \frac{2l}{d\pi}\left[\sin(\theta)|_0^{\pi/4} - \cos(\theta)|_{\pi/4}^{\pi/2}\right] = \frac{2l}{d\pi}\left[\sin(\pi/4) - \sin(0) - \cos(\pi/2) + \cos(\pi/4)\right]$$

$$= \frac{2l}{d\pi}\left(\frac{\sqrt{2}}{2} - 0 - 0 + \frac{\sqrt{2}}{2}\right) = \frac{2l}{d\pi}(\sqrt{2}) = \frac{2\sqrt{2}l}{d\pi}$$

For sake of brevity, I will not go over the case in which $l > d$. Also note that when $l \geq \sqrt{2}d$, the probability is 100% because there is no location where the needle can fall where it does not cross a line.

The Buffon-Laplace Problem

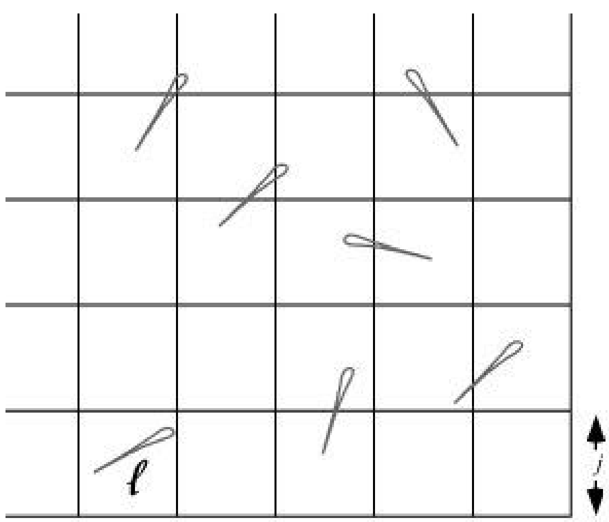


Figure 10

The necessary extension of the previous problem is one where the needle is dropped onto rectangles instead of squares. This scenario is referred to as the Buffon-Laplace problem as it was posed by Buffon and solved by French mathematician Pierre-Simon Laplace.[10] We will define the two side lengths of the rectangle as *j* and *k*.

[10] Wolfram Research, Buffon-Laplace Needle Problem, accessed September 28, 2020, https://mathworld.wolfram.com/Buffon-LaplaceNeedleProblem.html.

In this scenario we consider 3 separate variables:

1. The vertical distance from the bottom of the rectangle.
2. The horizontal distance from the left side of the rectangle
3. The angle at which the needle falls (θ).

Thus, the region where the center of the needle can land can be represented through the area of the rectangle.

$$A = (k - lcos(\theta))(j \pm lsin(\theta)) = kj \pm lksin(\theta) - jklcos(\theta) \ \pm l^2 cos(\theta)sin(\theta)$$

$$= jk - l(jcos(\theta) \pm ksin(\theta)) = jk - jlcos(\theta) - lk\,|sin(\theta)| + \tfrac{l^2}{2}\,|sin(2\theta)|$$

We take the angles $-\pi/2 \leq \theta \leq \pi/2$ because we no longer have the symmetry that was present in the previous problems. Thus, the probability can be computed by subtracting the integral of the above function from one.

$$p = 1 - \int_{-\pi/2}^{\pi/2} (jk - jlcos(\theta) - lk\,|sin(\theta)| + \tfrac{l^2}{2}\,|sin(2\theta)|)d\theta$$

$$= 1 - \left[jk\theta - jlsin(\theta) + jkcos(\theta) - l^2 sin(2\theta)\right] \Big|_{-\pi/2}^{\pi/2}$$

$$= 1 - \left(2lj + 2lk - l^2\right)/\pi jk \ = \ 1 - \tfrac{2l(j+k)-l^2}{\pi jk}$$

Hence, the complementary probability[11] is $\frac{2l(a+b)-l^2}{\pi jk}$. This seems like an overly complex solution, but makes perfect sense when we extend one side out to infinity, recreating the original Buffon’s Needle problem.

$$\lim_{j\to\infty}\left(\tfrac{2l(j+k)-l^2}{\pi jk}\right) = \lim_{j\to\infty}\left(\tfrac{2lj+2lk-l^2}{\pi jk}\right) = \lim_{j\to\infty}\left(\tfrac{2l\infty+2lk-l^2}{\pi\infty k}\right) = \tfrac{\infty}{\infty}$$

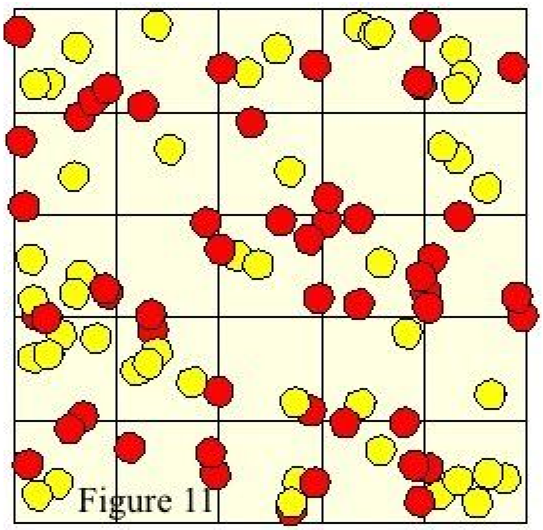
[11] Uspensky, *Introduction to Mathematical*, 256.

Now use L'Hopital's Rule because we have an indeterminate form, taking the derivative of the top and bottom of the fraction.

$$\lim_{j\to\infty}\left(\frac{2lj+2lk-l^2}{\pi jk}\right) = \lim_{j\to\infty}\left(\frac{2lj}{\pi jk}\right) = \frac{2l}{\pi k}$$

We have verified our solution to the original problem through an entirely different one!

Clean Tile Problems



Buffon also explored cases where coins (circularly shaped) were dropped onto planes of repeating polygons. This has been referred to as the Clean Tile Problem.[12] Let us first analyze the scenario we just solved, but for circles rather than needles. We define the diameter of the circle as *d* and the side length of the squares to be *l*.

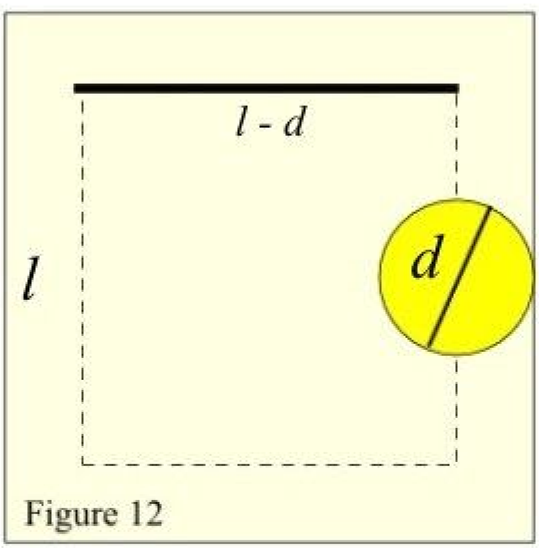


Figure 12

As shown in figure 12, The probability that a coin of diameter *d* will not cross any lines is:

$$\frac{(l-d)^2}{l^2} = \frac{l^2-2ld-d^2}{l^2} = \left(1-\frac{d}{l}\right)^2$$

Thus, the probability that a coin *will* land on a line is

$$1-\left(1-\frac{d}{l}\right)^2$$

Note that we don't have to incorporate any trigonometry in this problem because of the circles'

[12] Wolfram Research, Inc., Clean Tile Problem, accessed October 5, 2020, https://mathworld.wolfram.com/CleanTileProblem.html.

perfect rotational symmetry. Essentially, the circle represents all possible angles where the needle could fall.

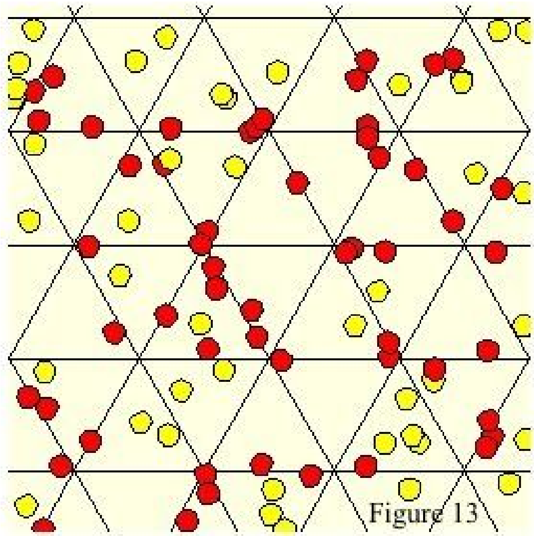


Figure 13

Let’s now consider the same scenario but with equilateral triangles of side length *d* (see figure 14). As shown in figure 14, the probability of a coin landing on no lines can be computed in the same way as above:

$$\frac{(1-\sqrt{3}d)^2}{l^2} = \frac{l^2-2\sqrt{3}dl-3d^2}{l^2} = \left(1-\frac{\sqrt{3}d}{l}\right)^2$$

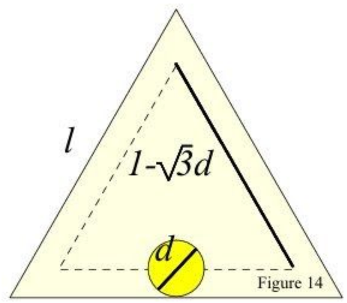


Figure 14

And follows the probability that a coins lands on a line:

$$1-\left(1-\frac{\sqrt{3}d}{l}\right)^2$$

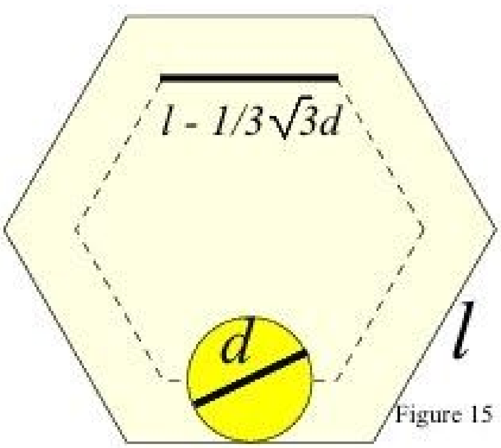
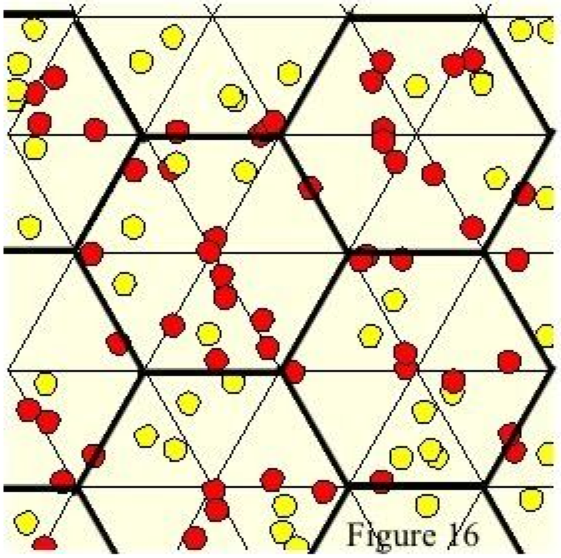


Figure 15

Next, let’s consider the situation where the plane is tiled with regular hexagons. We can utilize the area of a regular hexagon area formula: $\frac{1}{2}3\sqrt{3}l^2$. using the formula, the probability that a coin lands on no lines is:

$$\frac{(1-\sqrt{3}d)^2}{l^2} = \left(1-\frac{\sqrt{3}d}{3l}\right)^2$$

And $1-\left(1-\frac{\sqrt{3}d}{3l}\right)^2$ that a coin lands on a line.



Notice that the answer is quite similar to that of the scenario where the plane was tiled with equilateral triangles. Figure 16 depicts how the two scenarios are related. The hexagon can be broken into six congruent triangles that make up the same probability when combined.

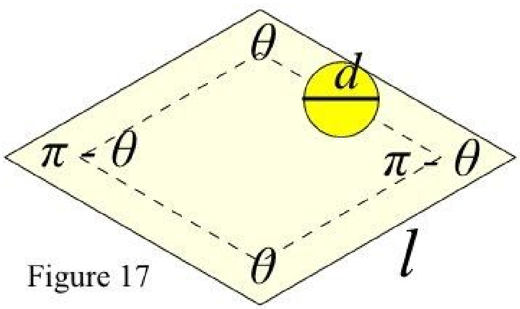


Figure 17

We will finish with a scenario involving a rhombus. This will be slightly more involved because of the introduction of a new variable. Because the internal angles of a rhombus can still vary even with a set side length *l*, we must take into account this variance.

We get $\Delta l_1 = \frac{1}{2} d\cot\theta$ and $\Delta l_2 = \frac{1}{2} d\tan\theta$

Thus

$$\Delta l = \frac{d}{2}(\cot\theta + \tan\theta) = \frac{d}{2}\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right) = \frac{d}{2}\left(\frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\sin\theta\cos\theta}\right) = \frac{d}{2}\left(\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}\right) = \frac{d}{2}\left(\frac{1}{\sin\theta\cos\theta}\right) = \frac{d}{2}\csc\theta\sec\theta$$

Thus the probability that a coin will not land on a line is

$$\frac{(l-\frac{d}{2}\csc\theta\sec\theta)^2}{l^2} = \left(1 - \frac{d}{2l}\csc\theta\sec\theta\right)^2 \text{ and}$$

$$1 - \left(1 - \frac{d}{2l}\csc\theta\sec\theta\right)^2 \text{ that it } \textit{will} \text{ land on a line.}$$

When $\theta = \frac{\pi}{4}$, the probability is $\left(1 - \frac{d}{2l}\csc\frac{\pi}{4}\sec\frac{\pi}{4}\right)^2 = \left(1 - \frac{d}{2l}\left(\frac{2}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right)\right)^2 = \left(1 - \frac{d}{2l}\left(\frac{4}{2}\right)\right)^2 = \left(1 - \frac{d}{l}\right)^2$,

which is the solution to the square case as expected.

Buffon's Ball Problem

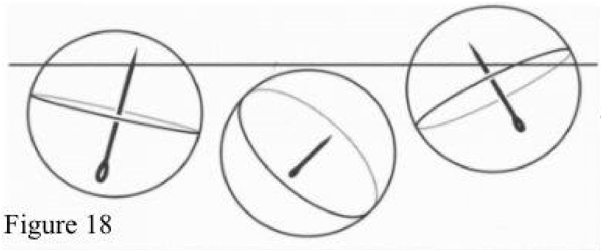


Figure 18

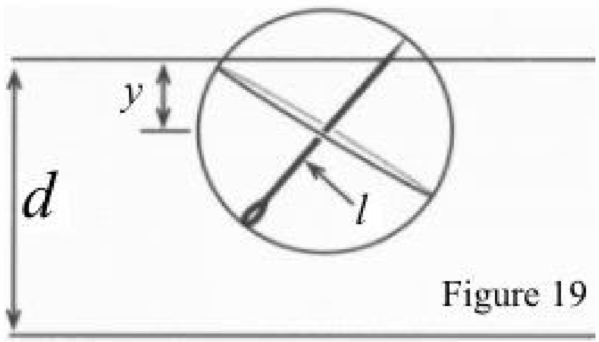


Figure 19

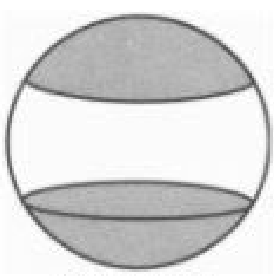


Figure 20

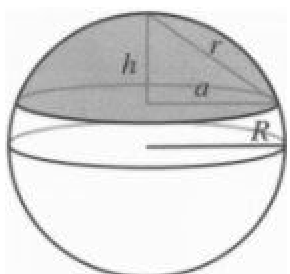


Figure 21

The last problem we will explore is called Buffon's Ball Problem.[13] Suppose that the needle has been placed into a transparent ball and is then dropped onto the plane containing parallel lines. The diameter of the ball will be denoted as l and the distance between the lines as d. This is equivalent to the probability that a needle is randomly placed in $\mathbf{R}^3$ and intersects a set of parallel planes separated by distance d. We will introduce another variable y, which is the distance of the base of the ball to the nearest line (See figure 19). Thus we are trying to find the probability that a needle crosses a line for a certain value of y. When $l/2 < y$, the needle cannot cross a line and when $0 \leq y \leq l/2$, the region of the sphere with needle tip locations that will produce a crossing can be represented by two symmetrical spherical 'caps' (See Figure 20). To find the area of these caps we use the formula $A = 2\pi Rh$, where h is the height of the cap and R is the radius of the sphere. Since the caps have height $l/2 - y$, the area of the regions of the sphere with needle tip locations that will produce a crossing is:

$$A = 2\pi Rh = 2\pi(\tfrac{l}{2})(\tfrac{l}{2} - y) = \pi l^2(1 - \tfrac{2y}{l})$$

[13] David Richeson, "A Pi-less Buffon's Needle Problem," *Mathematics Magazine* 79, no. 5 (December 2006): 385-387, https://doi.org/10.2307/27642977.

Thus, on $0 \leq y \leq l/2$, the probability that a needle in a ball crosses a line is the probability we just calculated divided by the total area of the sphere:

$$\frac{\pi l^2(1-\frac{2y}{l})}{\pi l^2} = 1 - \frac{2y}{l}$$

Notice how π is not even present in the solution! This is because finding the solution required no variable for an angle. How interesting! We seemingly added complexity to the problem by extending it into a 3rd dimension, but our answer is arguably even simpler!

Significance in the World of Math

Buffon's Needle was the first recorded usage of a Monte-Carlo method, which utilizes a random sampling technique to gauge probability instead of using direct computation. A Monte-Carlo method is most often utilized when a direct computation is not possible/extremely difficult and gained popularity when computers became powerful enough to run computations on a large-scale. Buffon's Needle was one of the primary Monte-Carlo methods used to approximate π before more efficient methods were discovered. The most notable example of this would be the approximation done by Mario Lazzarini in 1901 that accurately calculated π to six decimal places after only 3408 drops of a needle.[14] This problem has also been seen as the "founder" of the geometric probability field in mathematics that has led to further problems within this field such as Bertrand's Paradox.[15] Since the 20th century, the field has been divided into two separate branches: Integral geometry and Stochastic geometry.

[14] Badger, "Lazzarini's Lucky," 84.
[15] Swarthmore Edu, "Buffon's Needle," Math Images, last modified March 12, 2012, accessed October 5, 2020, https://mathimages.swarthmore.edu/index.php/Buffon%27s_Needle#Why_It.27s_Interesting.

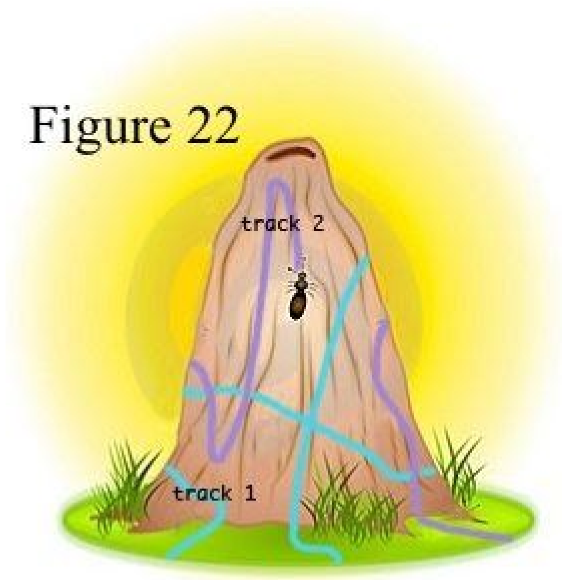


Figure 22

Researchers at the University of Bath found an especially interesting application of Buffon's Needle. Their study found that ants can accurately estimate the size of an anthill by visiting the hill twice and analyzing how many times their paths cross.[16] The ants enter the hill and walk around leaving behind a distinct chemical in their path. They later return to the hill and gauge how many times they cross their original path. More crosses mean a smaller hill and fewer crosses mean a larger hill. In summary, the study said: "In effect, an ant scout applies a variant of Buffon's needle theorem: The estimated area of a flat surface is inversely proportional to the number of intersections between the set of lines randomly scattered across the surface."[17] It is easiest to understand this application by noting that the ant is essentially running Buffon's Needle problem by moving the lines further apart. The larger the space between the lines, the lower the chance of intersection. As was shown in this investigation, the angle at which the needle falls makes a smaller difference when the space between the lines is longer (or the needle being shorter, both are equivalent statements). It's always awesome to see mathematics describing our natural world in a brilliant way such as this!

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[16] S. T. Mugford, E. B. Mallon, and N. R. Franks, "The Accuracy of Buffon's Needle: A Rule of Thumb Used by Ants to Estimate Area," *Behavioral Ecology* 12, no. 6 (November 2001): 655-658, https://doi.org/10.1093/beheco/12.6.655.

[17] Mugford, Mallon, and Franks, "The Accuracy," 655.

Buffon's Needle has led to the creation of a new field of mathematics (geometric probability) and has paved the way for further developments in the use of Monte-Carlo methods to approximate irrational numbers. As with most mathematical discoveries, Buffon had no way of knowing the significance of what seemed to him, a trivial mathematical question. He contributed to the world of mathematics in the best way he knew—by doing maths simply for the fun of it. His work goes to show that mathematics sees the most progress when its authors are not there on purpose, but by accident, intrigue, and sheer brilliance.

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