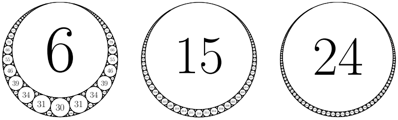
The Local-Global Conjecture for Apollonian Circle Packings

Summer Haag, **Clyde Kertzer**, James Rickards,
Katherine E. Stange

University of Colorado Boulder



(June 15, 2024)

Descartes Quadruples

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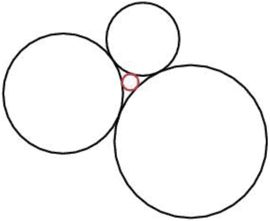

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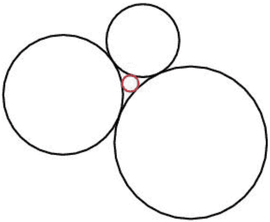

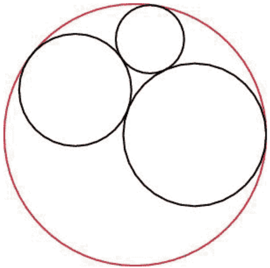

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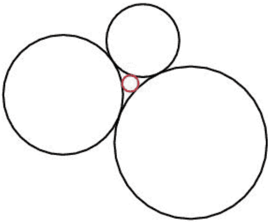

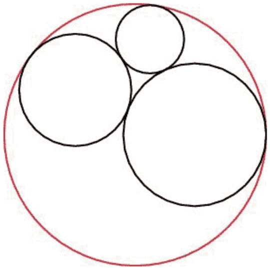

We can only have at most one "inverted" circle!

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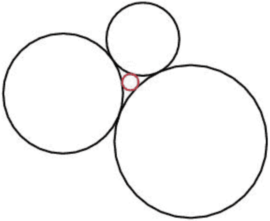

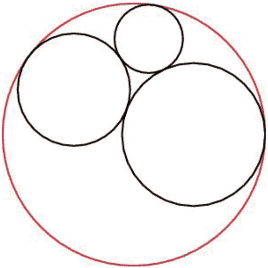

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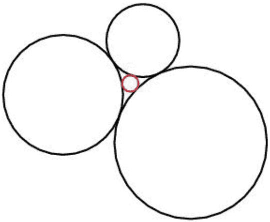

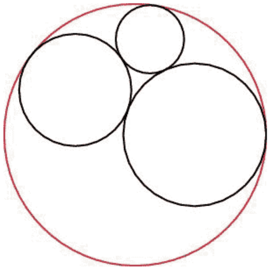

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Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

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The *curvature* of a circle with radius r is defined to be $1{=}r$.

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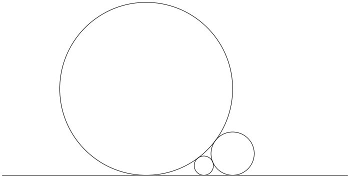

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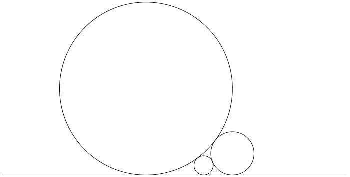

Circle with infinite radius!

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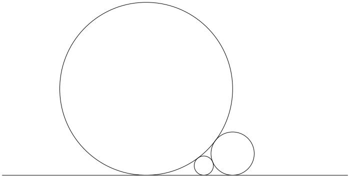

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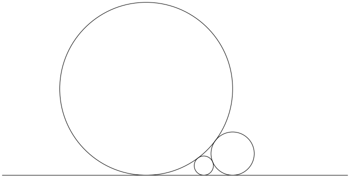

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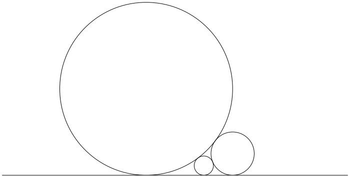

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Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2):$$

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If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

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$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

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If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Moreover, $d + d' = 2(a + b + c)$.

Apollonian Circle Packings

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The Key Relation

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$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

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$$d + d^0 = 2(a + b + c) \;\; =) \quad d^0 = 2(a + b + c) \quad\; d$$

If $a; b; c; d$ are integers, then d^0 is an integer!

Apollonian Circle Packings

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Apollonian Circle Packings

[1]Images from: AMS "When Kissing Involves Trigonometry"

Apollonian Circle Packings

$[-6, 11, 14, 23]$ [1]

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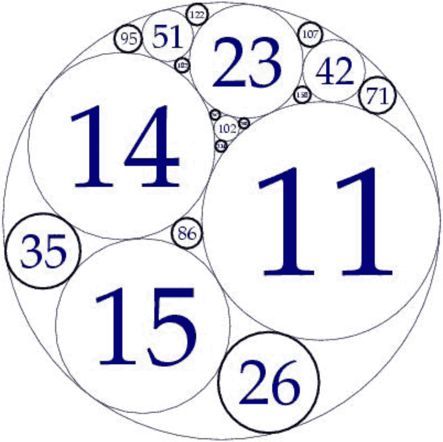
Apollonian Circle Packings

$[\quad 6; 11; 14; 23]$

Apollonian Circle Packings

[6; 11; 14; 23] reduces to [6; 11; 14; 15]

Apollonian Circle Packings



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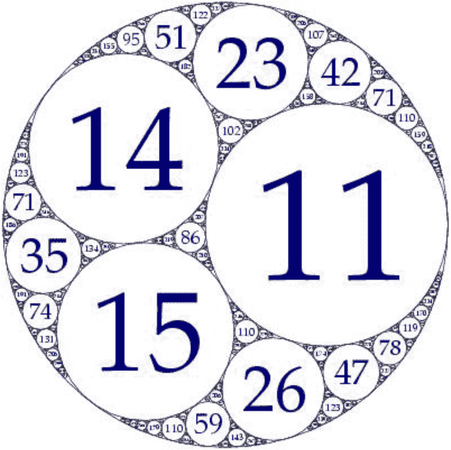

[6; 11; 14; 15]

Quadratic Forms

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Binary quadratic form (BQF)

$$Q(x, y) = Ax^2 + Bxy + Cy^2 = [A, B, C]$$

with $A, B, C \in \mathbb{Z}$

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Binary quadratic form (BQF)

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Fixing a circle a, the values of $f_a(x;y) \quad a$ with $\gcd(x;y) = 1$, a primitive integral binary quadratic form, are curvatures of circles tangent to a (Sarnak, Graham-Lagarias-Mallows-Wilks-Yan)

Curvatures Mod 5

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Curvatures Mod 3

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Admissible Residues

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$[\ \ 7; 12; 17; 20]$ $[\ \ 8; 13; 21; 24]$

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residues mod 24
0,1,4,9,12,16
0,5,8,12,20,21
0,4,12,13,16,21
0,8,9,12,17,20
3,6,7,10,15,18,19,22
2,3,6,11,14,15,18,23

Admissible Residues

Type	residues mod 24
(6,1)	0,1,4,9,12,16
(6,5)	0,5,8,12,20,21
(6,13)	0,4,12,13,16,21
(6,17)	0,8,9,12,17,20
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Theorem (Fuchs)

If a congruence obstruction appears, then it appears modulo 24.

Local-to-global

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[redacted]

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all su ciently large integers satisfying this condition appear.

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$[\ \ 7; 12; 17; 20]$ type $(6; 17) \Rightarrow$ $0; 8; 9; 12; 17; 20:$ no room for 8,9,32,...

History

$$K(N) := \{ n \leq N : n \text{ is a curvature} \}$$

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Number of circles of curvatures less than T grows like $T^{\delta + o(1)}$ with $\delta = 1.30\ldots$ = Hausdorff dim (Boyd, McMullen, Kontorovich-Oh)

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$K(N) \gg \sqrt{N}$ (Graham-Lagarias-Mallows-Wilks-Yan)

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- $K(N) \gg \frac{N}{\sqrt{\log(N)}}$ (Sarnak)

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- $K(N) \gg N$ (positive density) (Bourgain-Fuchs)
- $\exists \eta > 0$, $K(N) = \kappa N + O(N^{1-\eta})$ (density 1) (Bourgain-Kontorovich)
- $\exists \eta > 0$, $K(N) = \kappa N + O(N^{1-\eta})$ for a larger class of packings (Fuchs-Stange-Zhang)

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$$5\ \ 10^8 \text{ for } [\ 11; 21; 24; 28]$$

For $[\ 11; 21; 24; 28]$, there were still a small number (up to $0{:}013\%$) of missing curvatures in the range $(4\ \ 10^8; 5\ \ 10^8)$ for residue classes $0; 4; 12; 16 \pmod{24}$

Summer 2023 REU

Fix a pair of curvatures and see what packings contain them

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For an acceptable pair, black dot if no packing contains them

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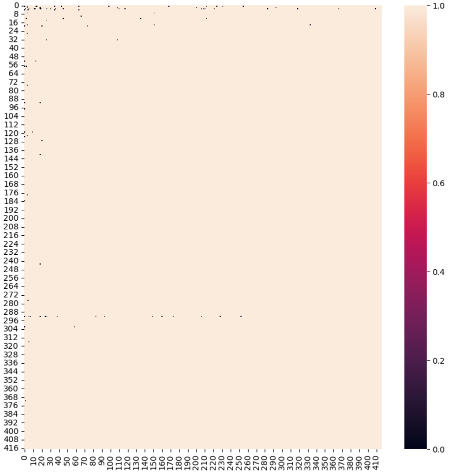
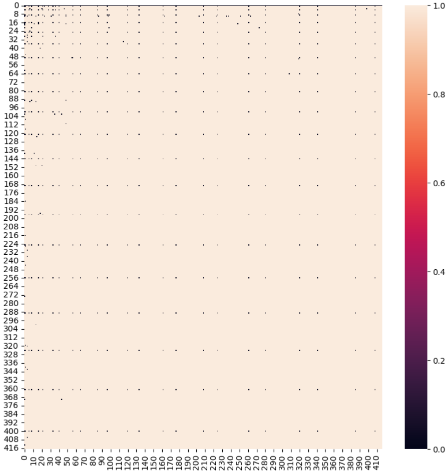
Local-to-global: fintely many black dots for a row or column

Usual Graph



Residue classes: 12 (mod 24) and 13 (mod 24)

Weird Graph



Residue classes: 0 (mod 24) and 8 (mod 24)

Local-to-global conjecture is false

(H.-K.-Rickards-Stange)

The Apollonian circle packing generated by $[\quad 3; 5; 8; 8]$ has no square curvatures.

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Note: $_2(C) = \frac{a}{n}$ for curvature a coprime and tangent to C

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Take $C_1; C_2$ tangent circles in ACP with curvatures $a; b$ respectively that are coprime

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Quadratic Reciprocity!!

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$$ {}_2(C_1) \; {}_2(C_2) = \quad \frac{a}{b} \quad \frac{b}{a} \quad = 1 \; =) \quad {}_2(C_1) = \; {}_2(C_2) $$

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$$ {}_2(C_1) \;\; {}_2(C_2) = \quad \frac{a}{b} \quad \frac{b}{a} \quad = 1 \implies \quad {}_2(C_1) = \quad {}_2(C_2) $$

Any two circles in ACP are connected by a path of pairwise coprime curvatures

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${}_2(C)$ is independent of choice of circle C!!

No Squares in [3; 5; 8; 8]

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Let A be the ACP generated by [3; 5; 8; 8], compute

$$_2(A) = \frac{8}{5} = \frac{3}{5} = 1$$

No Squares in $[\ 3; 5; 8; 8]$

Let A be the ACP generated by $[\ 3; 5; 8; 8]$, compute

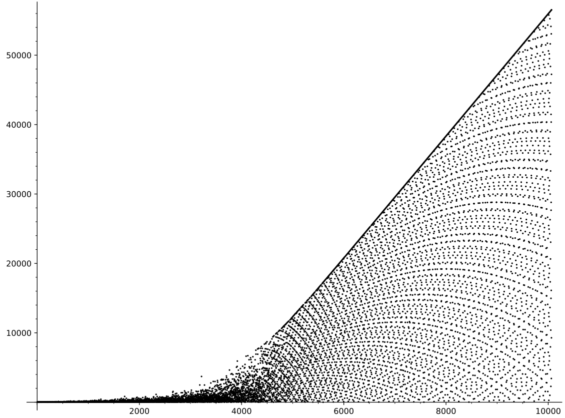
$$_2(A) = \quad \frac{8}{5} \quad = \quad \frac{3}{5} \quad = \quad 1$$

No circle can be tangent to a square

The New Conjecture

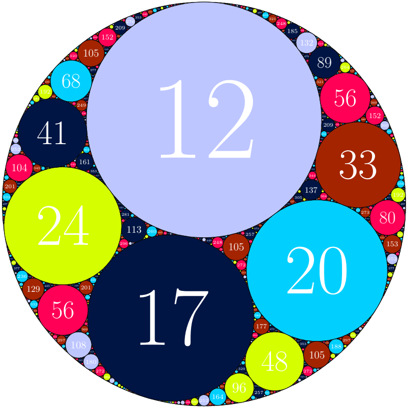
Type	Quadratic	Quartic	L-G false	L-G open
(6; 1; 1; 1)				0; 1; 4; 9; 12; 16
(6; 1; 1; 1)		$n^4; 4n^4; 9n^4; 36n^4$	0; 1; 4; 9; 12; 16	
(6; 1; 1)	$n^2; 2n^2; 3n^2; 6n^2$		0; 1; 4; 9; 12; 16	
(6; 5; 1)	$2n^2; 3n^2$		0; 8; 12	5; 20; 21
(6; 5; 1)	$n^2; 6n^2$		0; 12	5; 8; 20; 21
(6; 13; 1)	$2n^2; 6n^2$		0	4; 12; 13; 16; 21
(6; 13; 1)	$n^2; 3n^2$		0; 4; 12; 16	13; 21
(6; 17; 1; 1)	$3n^2; 6n^2$	$9n^4; 36n^4$	0; 9; 12	8; 17; 20
(6; 17; 1; 1)	$3n^2; 6n^2$	$n^4; 4n^4$	0; 9; 12	8; 17; 20
(6; 17; 1)	$n^2; 2n^2$		0; 8; 9; 12	17; 20
(8; 7; 1)	$3n^2; 6n^2$		3; 6	7; 10; 15; 18; 19; 22
(8; 7; 1)	$2n^2$		18	3; 6; 7; 10; 15; 19; 22
(8; 11; 1)				2; 3; 6; 11; 14; 15; 17; 23
(8; 11; 1)	$2n^2; 3n^2; 6n^2$		2; 3; 6; 18	11; 14; 15; 23

differences between successive missing curvatures



Successive differences of missing curvatures in the packing $(\ 4; 5; 20; 21)$. The quadratic families $2n^2$ and $3n^2$ begin to predominate (the sporadic set has 3659 elements $< 10^{10}$, and occur increasingly sparsely.)

Thank You!!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