The Local-Global Conjecture for Apollonian Circle Packings

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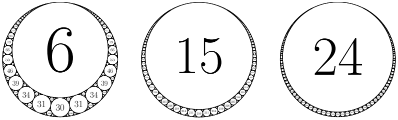
Katherine E. Stange

University of Colorado Boulder



(June 15, 2024)

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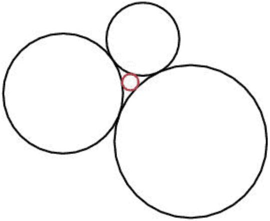

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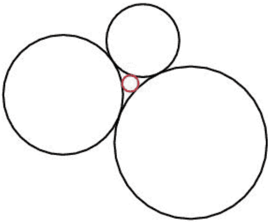

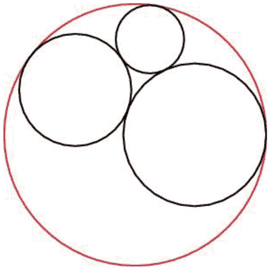

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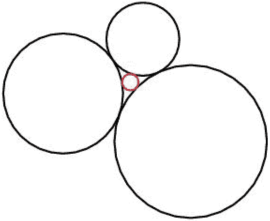

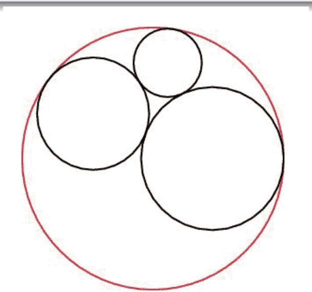

We can only have at most one "inverted" circle!

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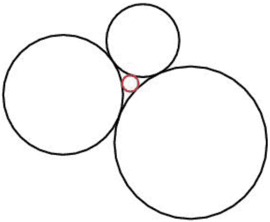

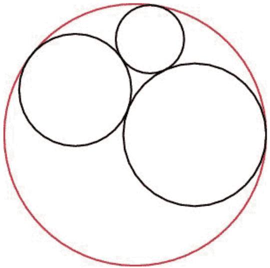

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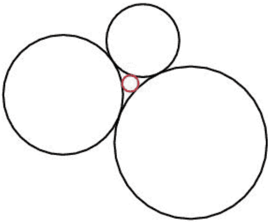

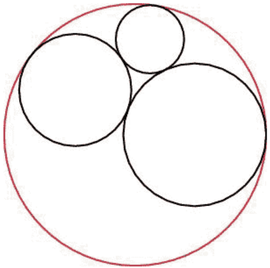

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Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

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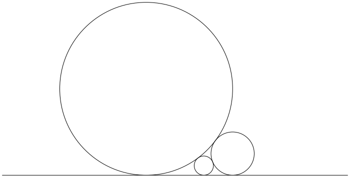

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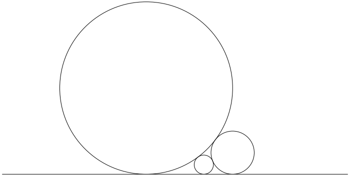

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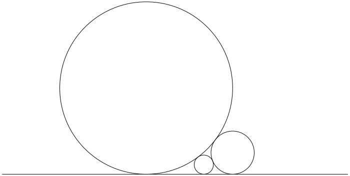

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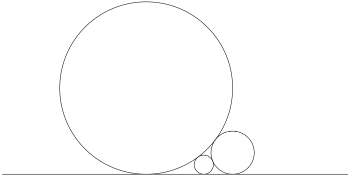

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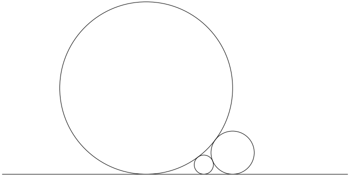

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If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

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Moreover, $d + d' = 2(a + b + c)$.

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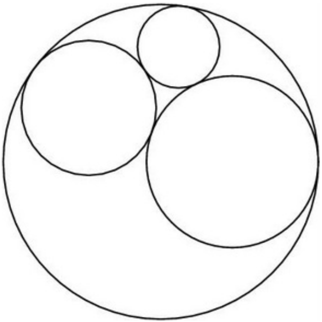

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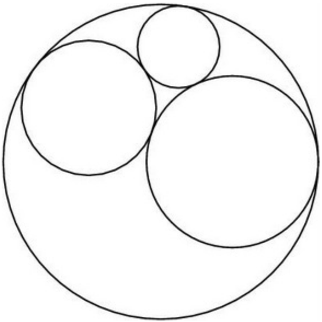

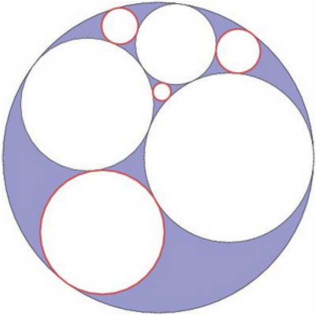

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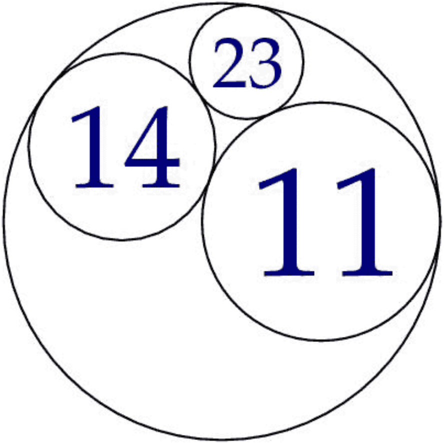

[1]Images from: AMS "When Kissing Involves Trigonometry"

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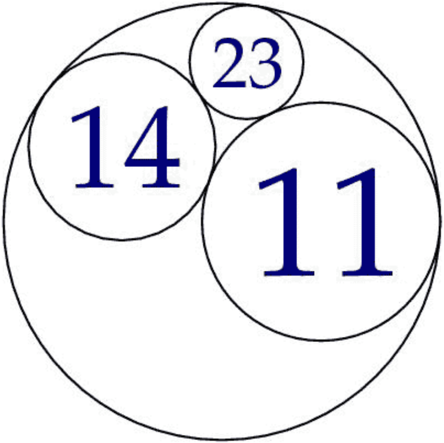

$[-6, 11, 14, 23]$ [1]

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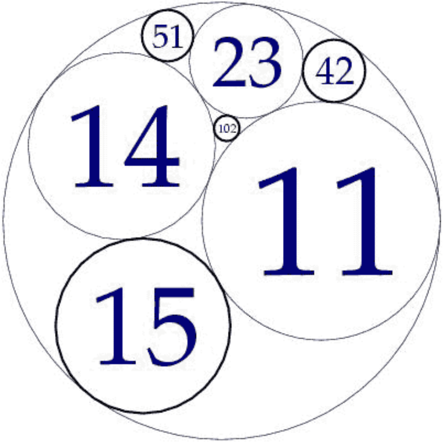

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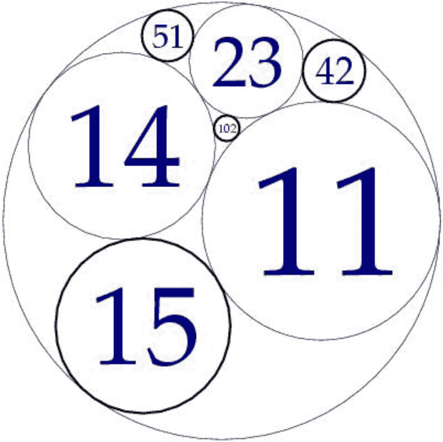
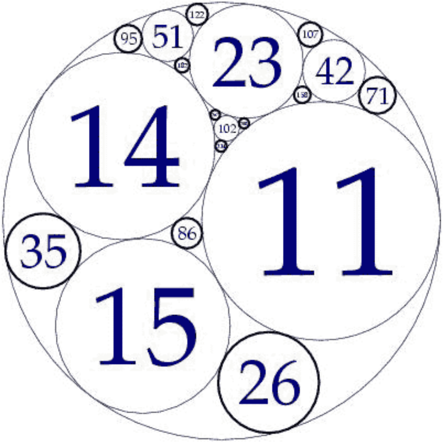


$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

Apollonian Circle Packings



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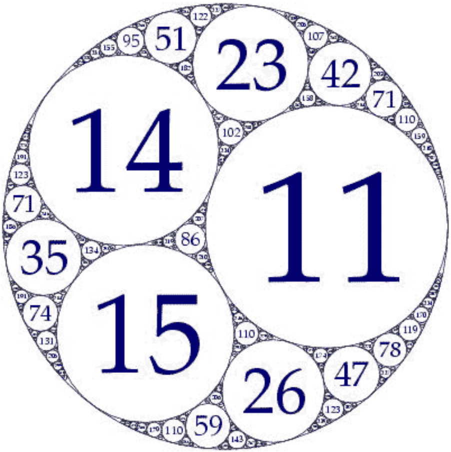

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Binary quadratic form (BQF)

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Fixing a circle a, the values of $f_a(x, y) - a$ with $\gcd(x, y) = 1$, a primitive integral binary quadratic form, are curvatures of circles tangent to a (Sarnak, Graham-Lagarias-Mallows-Wilks-Yan)

Curvatures Mod 5

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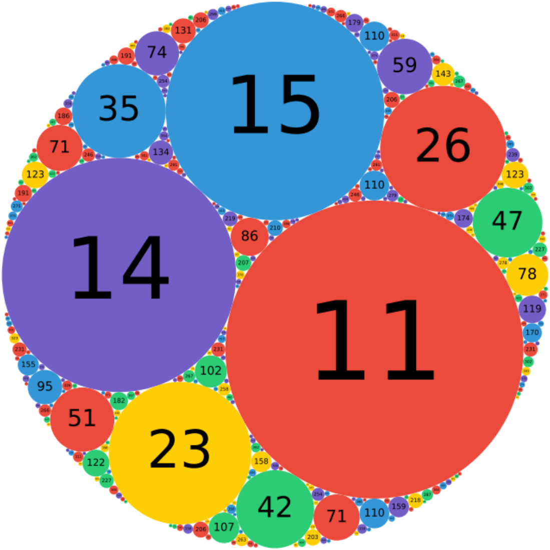

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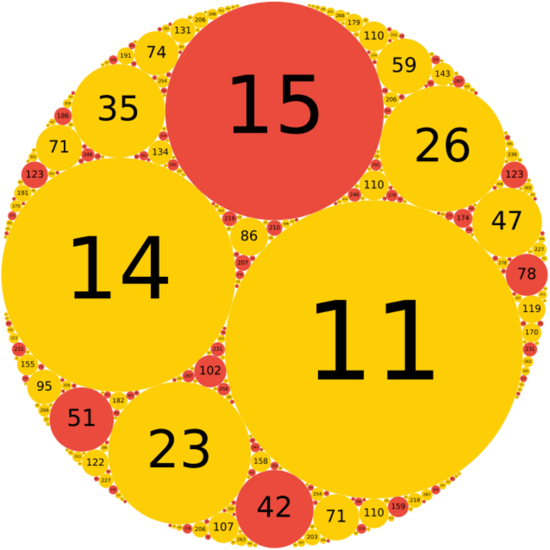

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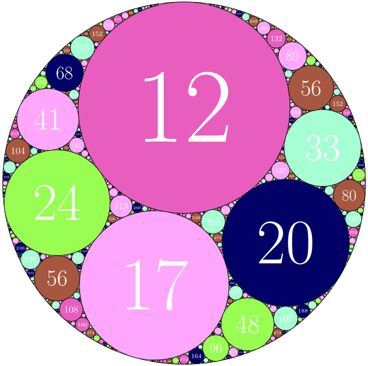
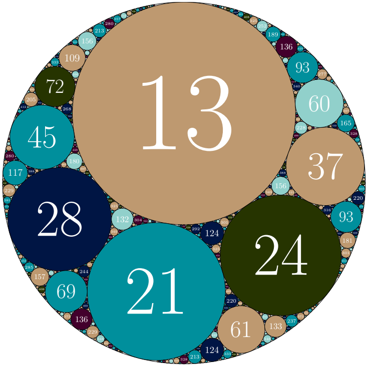


$[-7, 12, 17, 20]$



$[-8, 13, 21, 24]$

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residues mod 24
0,1,4,9,12,16
0,5,8,12,20,21
0,4,12,13,16,21
0,8,9,12,17,20
3,6,7,10,15,18,19,22
2,3,6,11,14,15,18,23

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Type	residues mod 24
(6,1)	0,1,4,9,12,16
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(6,13)	0,4,12,13,16,21
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Theorem (Fuchs)

If a congruence obstruction appears, then it appears modulo 24.

Local-to-global

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The Local-Global Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan '03, Fuchs-Sanden '11)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

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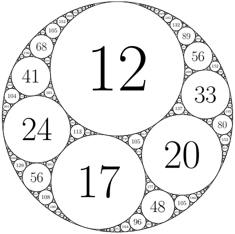

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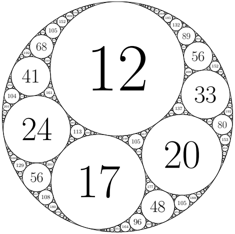

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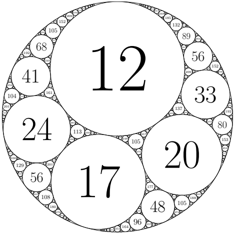

$[-7, 12, 17, 20]$ type $(6, 17) \implies 0, 8, 9, 12, 17, 20$: no room for 8,9,32,...

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- $\mathcal{K}(N) \gg N$ (positive density) (Bourgain-Fuchs)
- $\exists \eta > 0$, $\mathcal{K}(N) = kN + O(N^{1-\eta})$ (density 1) (Bourgain-Kontorovich)
- $\exists \eta > 0$, $\mathcal{K}(N) = kN + O(N^{1-\eta})$ for a larger class of packings (Fuchs-Stange-Zhang)

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For $[-11, 21, 24, 28]$, there were still a small number (up to 0.013%) of missing curvatures in the range $(4 \cdot 10^8, 5 \cdot 10^8)$ for residue classes $0, 4, 12, 16 \pmod{24}$

Summer 2023 REU

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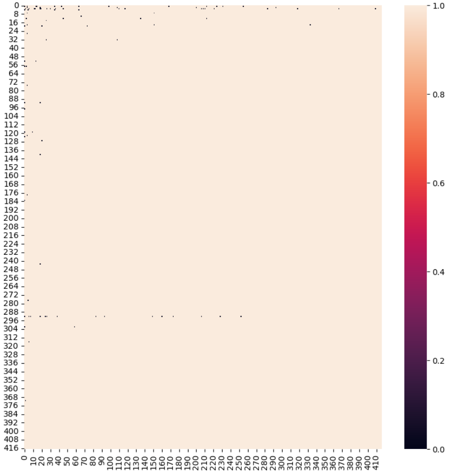
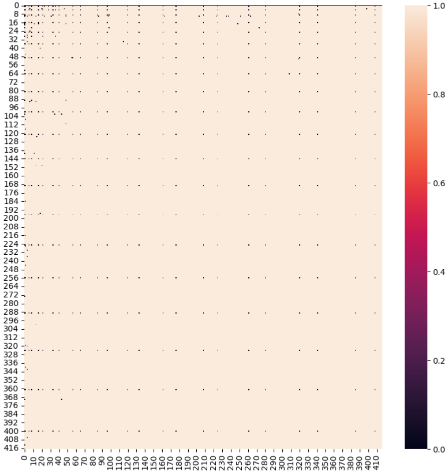
- Local-to-global: fintely many black dots for a row or column

Usual Graph



Residue classes: 12 (mod 24) and 13 (mod 24)

Weird Graph



Residue classes: 0 (mod 24) and 8 (mod 24)

Local-to-global conjecture is false

(H.-K.-Rickards-Stange)

The Apollonian circle packing generated by $[-3, 5, 8, 8]$ has no square curvatures.

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6. Note: $\chi_2(C) = \left(\frac{a}{n}\right)$ for curvature a coprime and tangent to C

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6. Note: $\chi_2(C) = \left(\frac{a}{n}\right)$ for curvature a coprime and tangent to C

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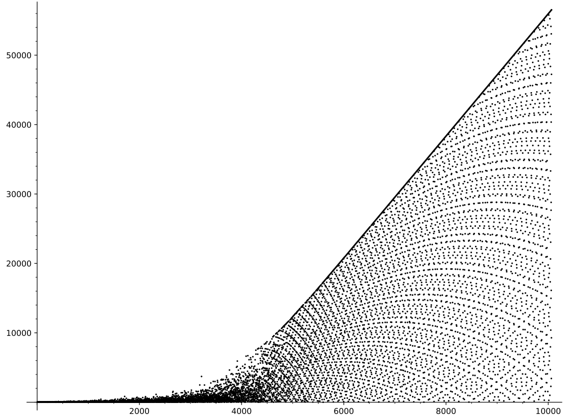
2. No circle can be tangent to a square

The New Conjecture

Type	Quadratic	Quartic	L-G false	L-G open
$(6,1,1,1)$				$0,1,4,9,12,16$
$(6,1,1,-1)$		$n^4, 4n^4, 9n^4, 36n^4$	$0,1,4,9,12,16$	
$(6,1,-1)$	$n^2, 2n^2, 3n^2, 6n^2$		$0,1,4,9,12,16$	
$(6,5,1)$	$2n^2, 3n^2$		$0,8,12$	$5,20,21$
$(6,5,-1)$	$n^2, 6n^2$		$0,12$	$5,8,20,21$
$(6,13,1)$	$2n^2, 6n^2$		0	$4,12,13,16,21$
$(6,13,-1)$	$n^2, 3n^2$		$0,4,12,16$	$13,21$
$(6,17,1,1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0,9,12$	$8,17,20$
$(6,17,1,-1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0,9,12$	$8,17,20$
$(6,17,-1)$	$n^2, 2n^2$		$0,8,9,12$	$17,20$
$(8,7,1)$	$3n^2, 6n^2$		$3,6$	$7,10,15,18,19,22$
$(8,7,-1)$	$2n^2$		18	$3,6,7,10,15,19,22$
$(8,11,1)$				$2,3,6,11,14,15,17,23$
$(8,11,-1)$	$2n^2, 3n^2, 6n^2$		$2,3,6,18$	$11,14,15,23$

differences between successive missing curvatures



Successive differences of missing curvatures in the packing $(-4, 5, 20, 21)$. The quadratic families $2n^2$ and $3n^2$ begin to predominate (the sporadic set has 3659 elements $< 10^{10}$, and occur increasingly sparsely.)

Thank You!!