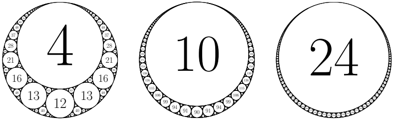
Apollonian Circle Packings & Parameterizations of Descartes Quadruples

Clyde Kertzer

University of Colorado Boulder



(Wednesday, April 10, 2024)

Descartes Quadruples

Descartes Quadruples

Definition

Descartes Quadruples

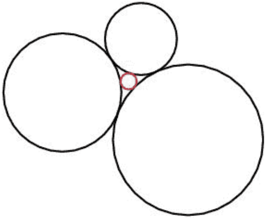
Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.

Descartes Quadruples

Definition

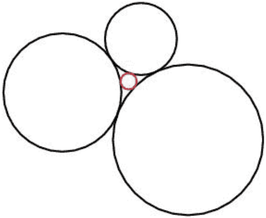
Descartes quadruple: four mutually tangent circles with disjoint interiors.

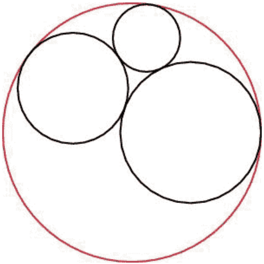


Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.

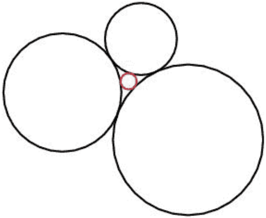


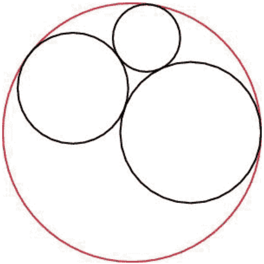


Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.



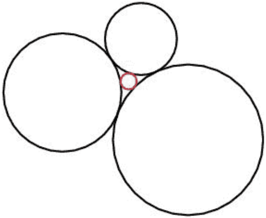


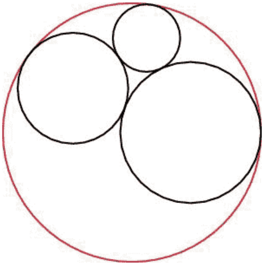
We can only have at most one "inverted" circle!

Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.





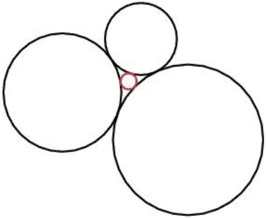
We can only have at most one "inverted" circle!

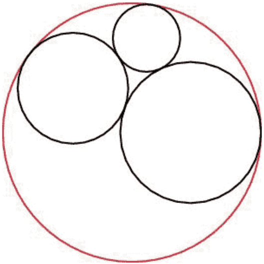
Theorem of Apollonius

Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.





We can only have at most one "inverted" circle!

Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

The Descartes Equation

The Descartes Equation

Definition

The Descartes Equation

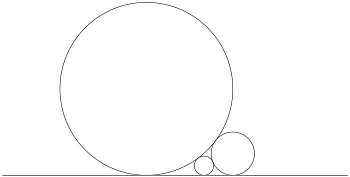
Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

The Descartes Equation

Definition

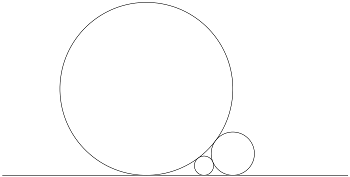
The *curvature* of a circle with radius r is defined to be $1/r$.



The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

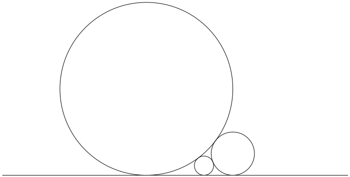


Circle with infinite radius

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



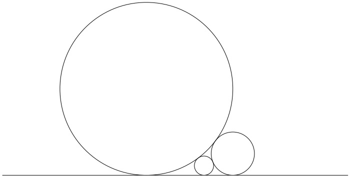
Circle with infinite radius

Descartes Equation

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius

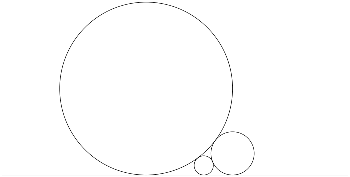
Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius

Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

The Descartes Equation

The Descartes Equation

Corollary

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d′ have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Moreover, $d + d' = 2(a + b + c)$.

The Descartes Equation

The Descartes Equation

Proof.

Proof.

First, we solve for d from the Descartes Equation to find that

The Descartes Equation

Proof.

First, we solve for d from the Descartes Equation to find that

$$2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2=0$$
$$d^2-2d(a+b+c)+\left(a^2+b^2+c^2-2ab-2bc-2ac\right)=0.$$

Proof.

First, we solve for d from the Descartes Equation to find that

$$2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2=0$$

$$d^2-2d(a+b+c)+\left(a^2+b^2+c^2-2ab-2bc-2ac\right)=0.$$

The quadratic formula gives

The Descartes Equation

Proof.

First, we solve for d from the Descartes Equation to find that

$$2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2=0$$

$$d^2-2d(a+b+c)+\left(a^2+b^2+c^2-2ab-2bc-2ac\right)=0.$$

The quadratic formula gives

$$\begin{aligned} d &= (a+b+c) \\ &\quad \pm \frac{\sqrt{4(a+b+c)^2-4\left(a^2+b^2+c^2-2ab-2bc-2ac\right)}}{2} \\ &= a+b+c \pm 2\sqrt{ab+bc+ca}. \end{aligned}$$

The Descartes Equation

Proof.

First, we solve for d from the Descartes Equation to find that

$$2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2=0$$
$$d^2-2d(a+b+c)+\left(a^2+b^2+c^2-2ab-2bc-2ac\right)=0.$$

The quadratic formula gives

$$\begin{aligned} d &= (a+b+c) \\ &\quad \pm \frac{\sqrt{4(a+b+c)^2-4\left(a^2+b^2+c^2-2ab-2bc-2ac\right)}}{2} \\ &= a+b+c\pm 2\sqrt{ab+bc+ca}. \end{aligned}$$

Thus, there are two options for d. Their sum is $2(a+b+c)$. □

Apollonian Circle Packings

The Key Relation

The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

The Key Relation

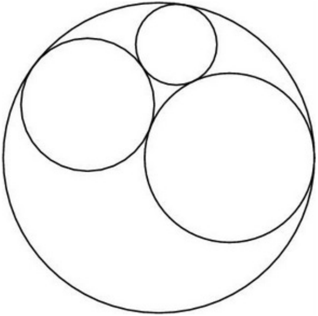
$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

If a, b, c, d are integers, then d' is an integer!

The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

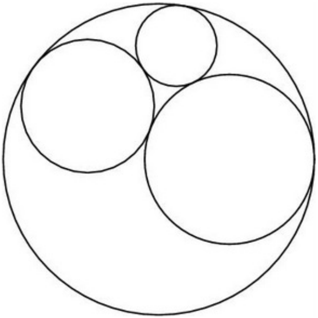
If a, b, c, d are integers, then d' is an integer!

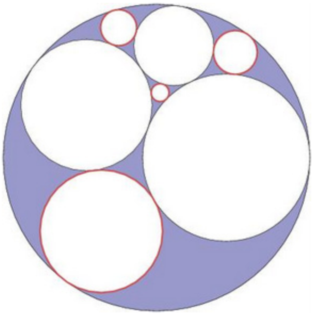


The Key Relation

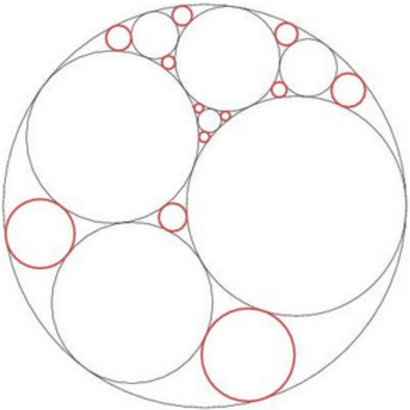
$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

If a, b, c, d are integers, then d' is an integer!

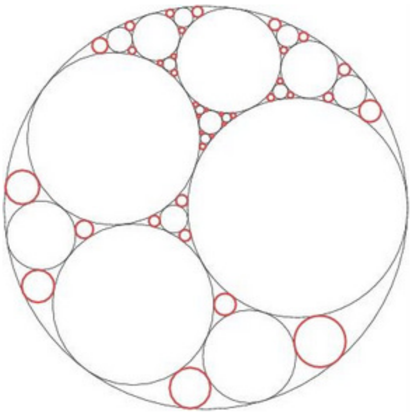




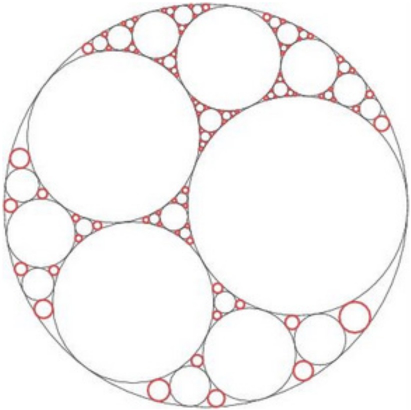
Apollonian Circle Packings



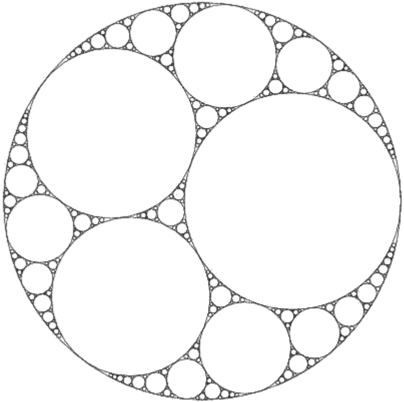
Apollonian Circle Packings

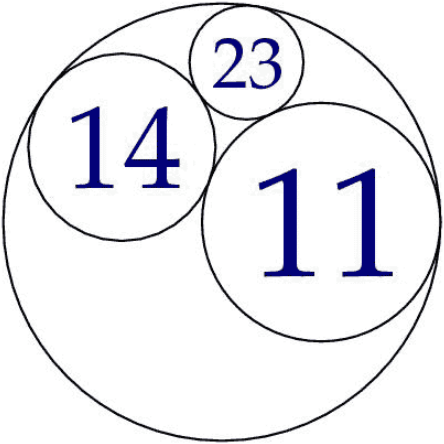


Apollonian Circle Packings

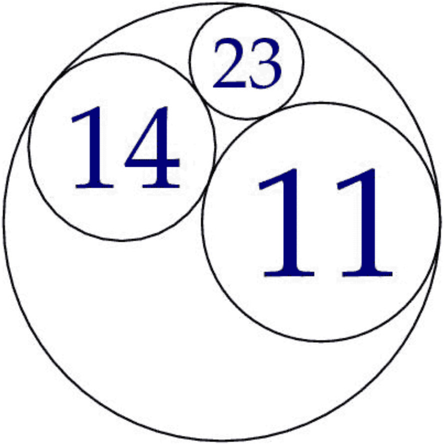


Apollonian Circle Packings





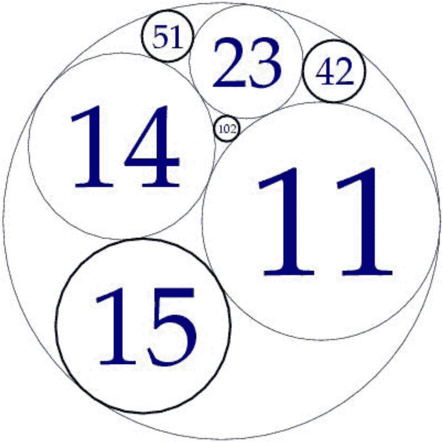
[1]Images from: AMS "When Kissing Involves Trigonometry"



$[-6, 11, 14, 23]$ [1]

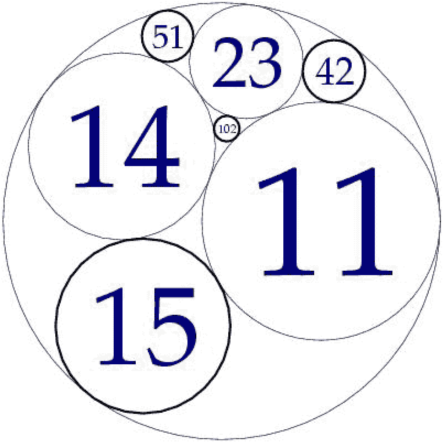
[1] Images from: AMS "When Kissing Involves Trigonometry"

Apollonian Circle Packings



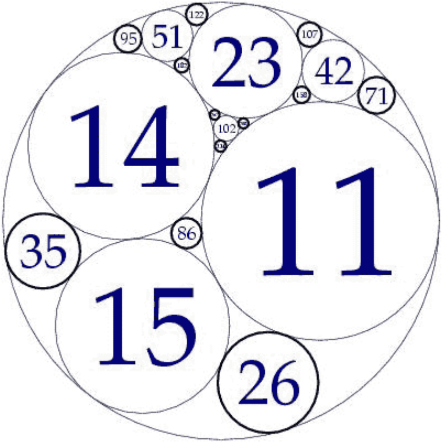
$[-6, 11, 14, 23]$

Apollonian Circle Packings

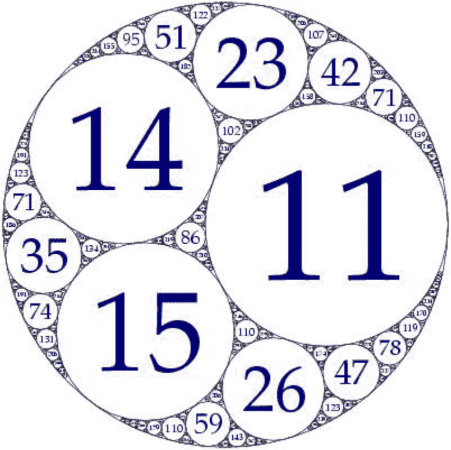


$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

Apollonian Circle Packings



$[-6, 11, 14, 15]$


11
14
15
23
26
35
42
47
51
59
71
71
74
78
86
95
102
107
110
110
110
119
122
123
123
134
143

$[-6, 11, 14, 15]$

Definition

Definition

A positive integer *a has a packing*

Definition

A positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

Definition

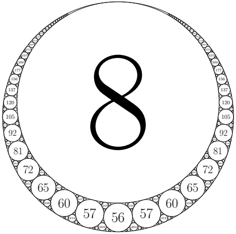
A positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

Example: $a = 7$

Definition

A positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

Example: $a = 7$

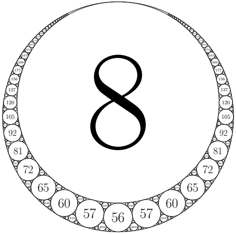


$[-7, 8, 56, 57]$,

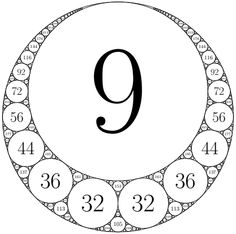
Definition

A positive integer a *has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

Example: $a = 7$



$[-7, 8, 56, 57]$,

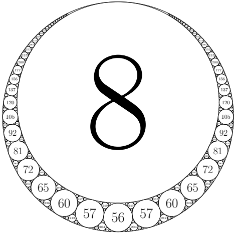


$[-7, 9, 32, 32]$,

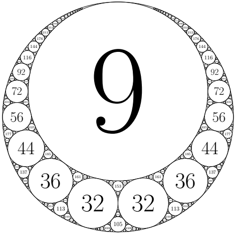
Definition

A positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

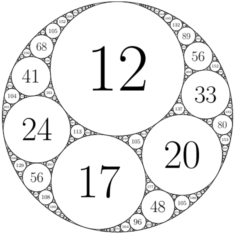
Example: $a = 7$



$[-7, 8, 56, 57]$,



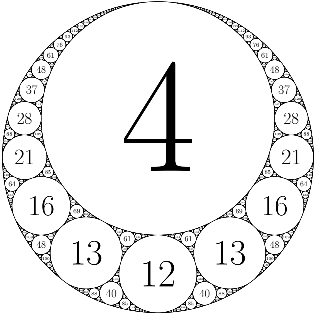
$[-7, 9, 32, 32]$,



$[-7, 12, 17, 20]$.

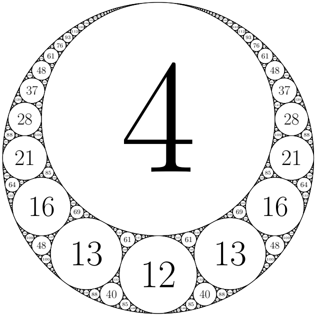
Symmetric Packings

Symmetric Packings

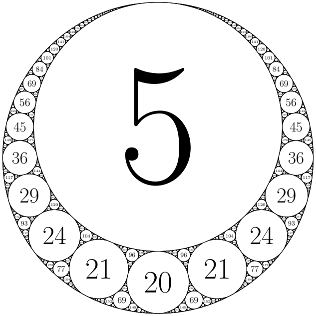


[−3, 4, 12, 13]

Symmetric Packings

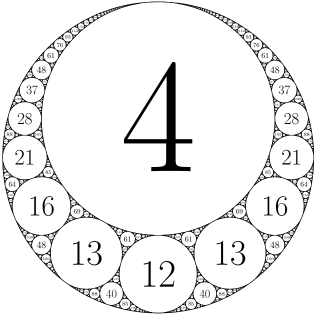


[−3, 4, 12, 13]

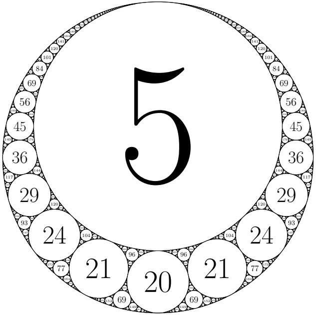


[−4, 5, 20, 21]

Symmetric Packings



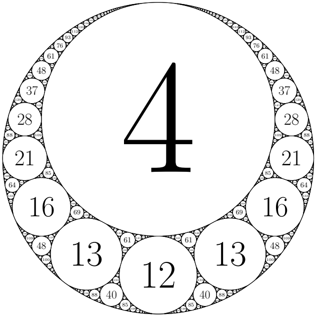
[−3, 4, 12, 13]



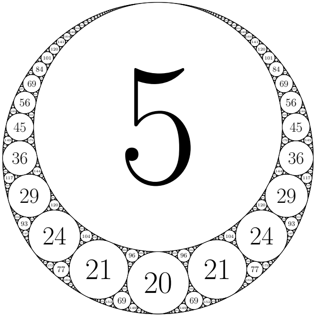
[−4, 5, 20, 21]

Definition

Symmetric Packings



[−3, 4, 12, 13]

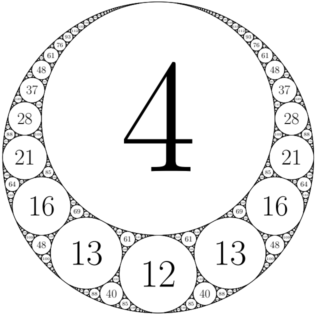


[−4, 5, 20, 21]

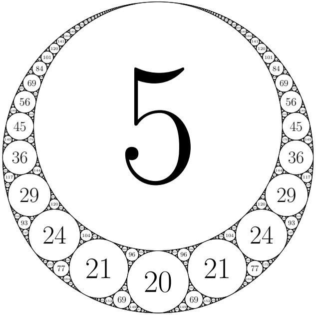
Definition

A *sum-symmetric*

Symmetric Packings



$[-3, 4, 12, 13]$

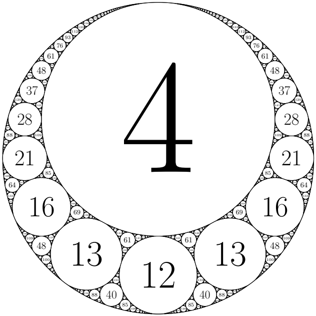


$[-4, 5, 20, 21]$

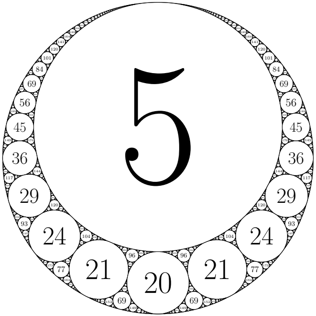
Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$.

Symmetric Packings



[−3, 4, 12, 13]



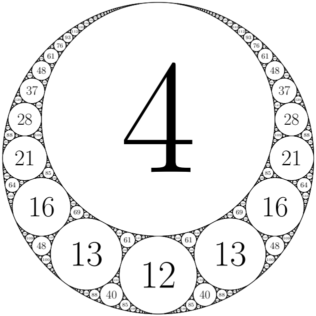
[−4, 5, 20, 21]

Definition

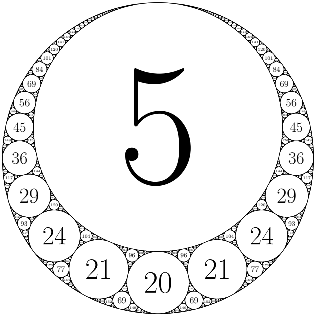
A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a+b+c)-d=d$.

$$2(a+b+c)-d=d$$

Symmetric Packings



$[-3, 4, 12, 13]$



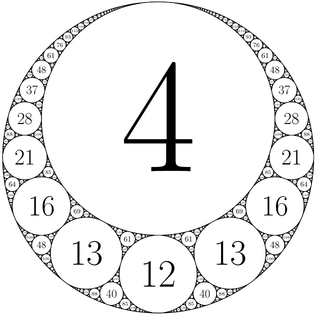
$[-4, 5, 20, 21]$

Definition

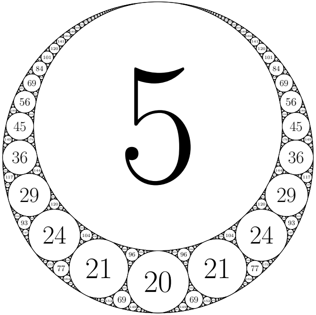
A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a+b+c)-d=d$.

$$2(a+b+c)-d=d \implies 2(a+b+c)=2d$$

Symmetric Packings



$[-3, 4, 12, 13]$



$[-4, 5, 20, 21]$

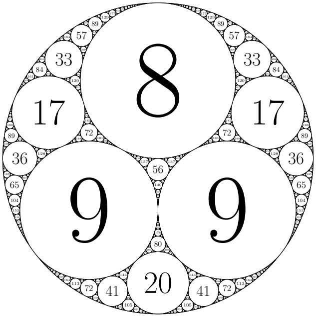
Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a+b+c)-d=d$.

$$2(a+b+c)-d=d \implies 2(a+b+c)=2d \implies a+b+c=d$$

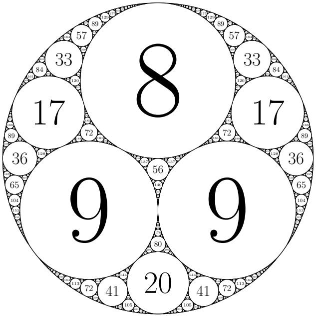
Symmetric Packings

Symmetric Packings

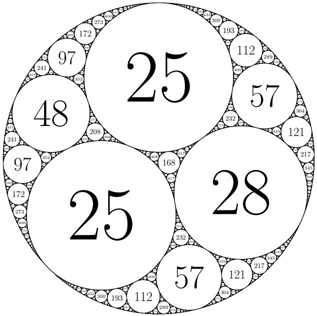


$[-4, 8, 9, 9]$

Symmetric Packings

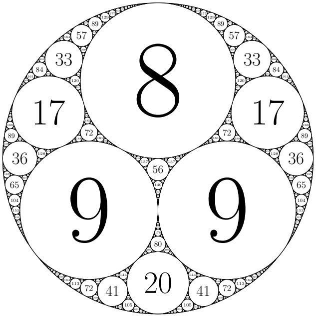


[−4, 8, 9, 9]

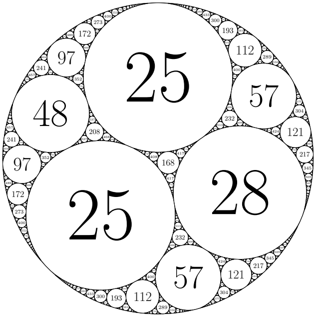


[−12, 25, 25, 28]

Symmetric Packings



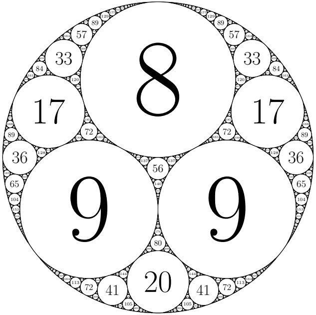
$[-4, 8, 9, 9]$



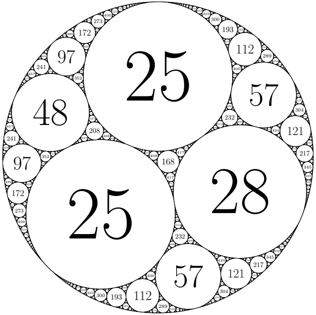
$[-12, 25, 25, 28]$

Definition

Symmetric Packings



[−4, 8, 9, 9]

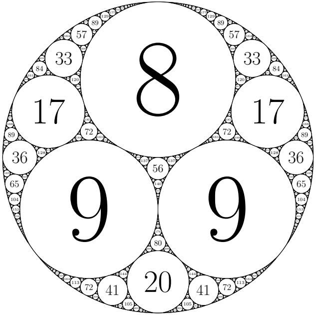


[−12, 25, 25, 28]

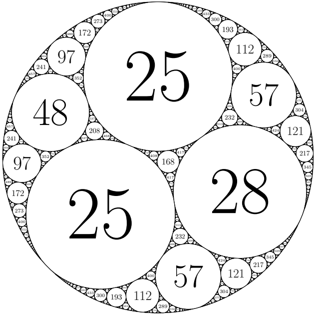
Definition

A *twin-symmetric* quadruple

Symmetric Packings



[−4, 8, 9, 9]



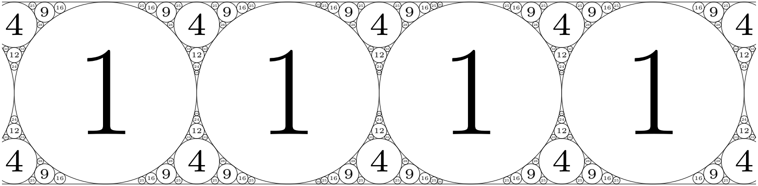
[−12, 25, 25, 28]

Definition

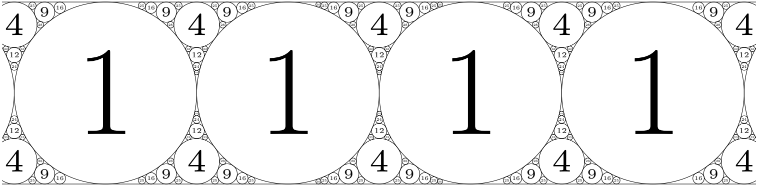
A *twin-symmetric* quadruple is a primitive reduced Descartes quadruple with $c = d$ or $c = b$.

The Two Unusual Symmetric Packings

The Two Unusual Symmetric Packings



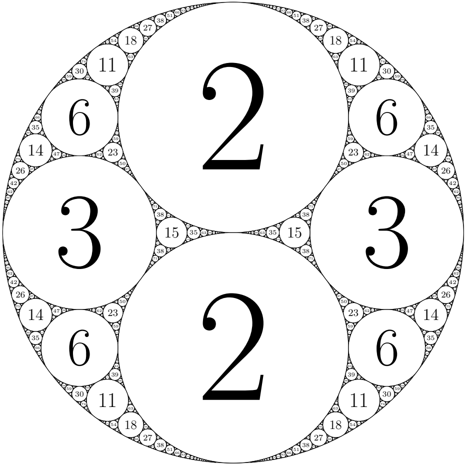
The Two Unusual Symmetric Packings



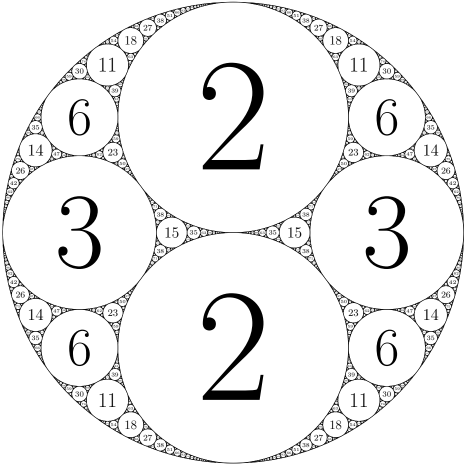
The strip packing: $[0, 0, 1, 1]$

The Two Unusual Symmetric Packings

The Two Unusual Symmetric Packings



The Two Unusual Symmetric Packings



The bug-eye packing: $[-1, 2, 2, 3]$

Symmetric Packings

Proposition

Symmetric Packings

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Symmetric Packings

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition

Symmetric Packings

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition

Only the strip and bug-eye packing are both sum-symmetric and twin-symmetric.

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$			

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$d-b$	$d+a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81
$[-21, 30, 70, 79]$	9	49	100

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2		3^2		5^2
$[-12, 21, 28, 37]$	3^2		4^2		7^2
$[-18, 22, 99, 103]$	2^2		9^2		11^2
$[-20, 36, 45, 61]$	4^2		5^2		9^2
$[-21, 30, 70, 79]$	3^2		7^2		10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$b+a$	$d-b$	$c+a$	$d+a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$b+a$	$d-b$	$c+a$	$d+a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2\cdot 3)}_{-6},\ \underbrace{2^2+2\cdot 3}_{10},\ \underbrace{3^2+2\cdot 3}_{15},\ \underbrace{(2+3)^2-2\cdot 3}_{19}\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$b+a$	$d-b$	$c+a$	$d+a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2\cdot 3)}_{-6},\ \underbrace{2^2+2\cdot 3}_{10},\ \underbrace{3^2+2\cdot 3}_{15},\ \underbrace{(2+3)^2-2\cdot 3}_{19}\right]$$

$$\left[-xy,\ x^2+xy,\ y^2+xy,\ (x+y)^2-xy\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$b+a$	$d-b$	$c+a$	$d+a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2\cdot 3)}_{-6},\ \underbrace{2^2+2\cdot 3}_{10},\ \underbrace{3^2+2\cdot 3}_{15},\ \underbrace{(2+3)^2-2\cdot 3}_{19}\right]$$

$$\left[-xy,\ x^2+xy,\ y^2+xy,\ (x+y)^2-xy\right]$$

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$d-b$	$d+a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13] \quad (x = 3, y = 1)$$

Sum-Symmetric Packings

Sum-Symmetric Packings

Theorem

Sum-Symmetric Packings

Theorem

A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

Sum-Symmetric Packings

Theorem

A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

Sum-Symmetric Packings

Theorem

A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

with $\gcd(x, y) = 1$*, and* $x, y \geqslant 0$*.*

The Number of Sum-Symmetric Packings

The Number of Sum-Symmetric Packings

Corollary

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y,

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy but we divide by two to account for symmetry.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy but we divide by two to account for symmetry. Thus, n has $2^k/2 = 2^{k-1} = 2^{\omega(n)-1}$ sum-symmetric packings. □

Sum-Symmetric packings of 60

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$,

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$.

Sum-Symmetric packings of 60

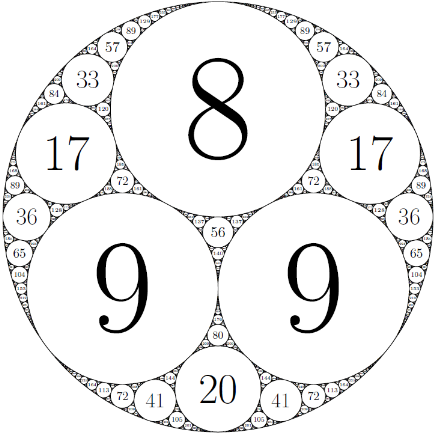
Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$. They are

$$\begin{aligned}
(1, 60) &\implies [-60, 61, 3660, 3661] \\
(4, 15) &\implies [-60, 76, 285, 301] \\
(3, 20) &\implies [-60, 69, 460, 469] \\
(5, 12) &\implies [-60, 85, 204, 229]
\end{aligned}$$

Twin-Symmetric Packings

Packings where one of the numbers is the same:

Packings where one of the numbers is the same:



$[-4, 8, 9, 9]$

Twin-Symmetric Packings

Twin-Symmetric Packings

-2	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$
-16	$[-16, 20, 81, 81]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$
-16	$[-16, 20, 81, 81]$

Twin-Symmetric Packings

Twin-Symmetric Packings

Over the summer:

Twin-Symmetric Packings

Over the summer:

Theorem

Twin-Symmetric Packings

Over the summer:

Theorem

All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

$$\left[-x,\ x+2y^2,\ 2\left(\frac{x+y^2}{2y}\right)^2,\ 2\left(\frac{x+y^2}{2y}\right)^2\right] \quad y \text{ odd}$$

Over the summer:

Theorem

All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

$$\left[-x,\ x+2y^2,\ 2\left(\frac{x+y^2}{2y}\right)^2,\ 2\left(\frac{x+y^2}{2y}\right)^2\right] \quad y \text{ odd}$$

Not ideal, not in terms of factorization.

Twin-Symmetric Packings

Improved to:

Improved to:

Theorem

Twin-Symmetric Packings

Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{(x+y)^2}{2},\ \frac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x > y \\ & \\ & \end{cases}$$

Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{(x+y)^2}{2},\ \frac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x>y \\ \left[-xy,\ xy+4y^2,\ \left(\frac{x}{2}+y\right)^2,\ \left(\frac{x}{2}+y\right)^2\right] & 4 \mid x, \quad x>2y \end{cases}$$

Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \dfrac{(x+y)^2}{2},\ \dfrac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x>y \\ \left[-xy,\ xy+4y^2,\ \left(\dfrac{x}{2}+y\right)^2,\ \left(\dfrac{x}{2}+y\right)^2\right] & 4 \mid x, \quad x>2y \\ \left[-xy,\ xy+x^2,\ \left(\dfrac{x}{2}+y\right)^2,\ \left(\dfrac{x}{2}+y\right)^2\right] & 4 \mid x, \quad x<2y \end{cases}$$

with $\gcd(x,y)=1$*.*

Twin-Symmetric Packings

Further improved to:

Further improved to:

Theorem

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Twin-Symmetric Packings

Further improved to:

Theorem

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x,y \text{ odd} \quad x>y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x>y \end{cases}$$

with $\gcd(x,y)=1$ *and* $x,y \geqslant 0$.

Ex: $x=3,\ y=2$

Further improved to:

Theorem

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Ex: $x = 3,\ y = 2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Further improved to:

Theorem

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Ex: $x = 3,\ y = 2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Why won't $x = 1, y = 3$ work?

Further improved to:

Theorem

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Ex: $x = 3,\ y = 2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Why won't $x = 1, y = 3$ work? Let's try:

Twin-Symmetric Packings

Further improved to:

Theorem

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Ex: $x = 3,\ y = 2$:

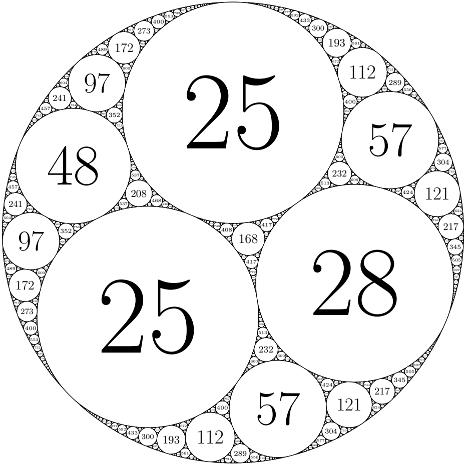
$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Why won't $x = 1, y = 3$ work? Let's try:

$$[-3, 3+2(3)^2, 5^2, 5^2] \implies [-12, 48, 25, 25]$$

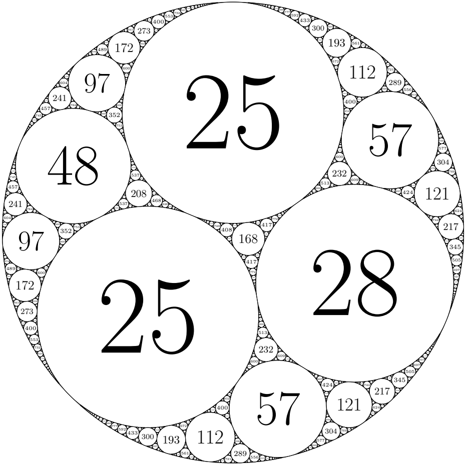
Twin-symmetric Packings

Twin-symmetric Packings



$[-12, 48, 25, 25]$

Twin-symmetric Packings



$$[-12, 48, 25, 25] \implies [-12, 28, 25, 25]$$

The Number of Twin-symmetric Packings

We define δ_n as

The Number of Twin-symmetric Packings

We define δ_n as

$$\delta_n = \begin{cases} 1 & n \equiv 2 \text{ mod } 4 \\ 0 & \text{otherwise.} \end{cases}$$

We define δ_n as

$$\delta_n = \begin{cases} 1 & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary

The Number of Twin-symmetric Packings

We define δ_n as

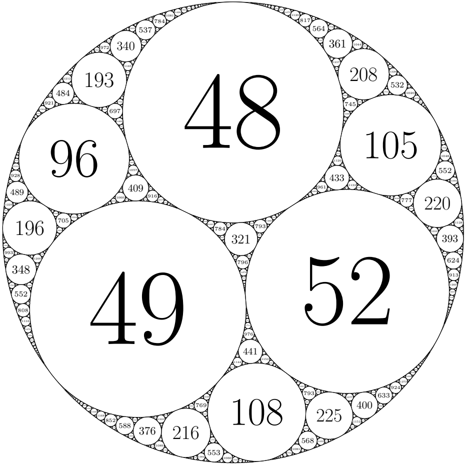
$$\delta_n = \begin{cases} 1 & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary

A natural number n has $(1 - \delta_n) \cdot 2^{\omega(n)-1}$ *twin-symmetric packings where* $\omega(n)$ *is the number of distinct prime divisors of n.*

Non-symmetric Packings

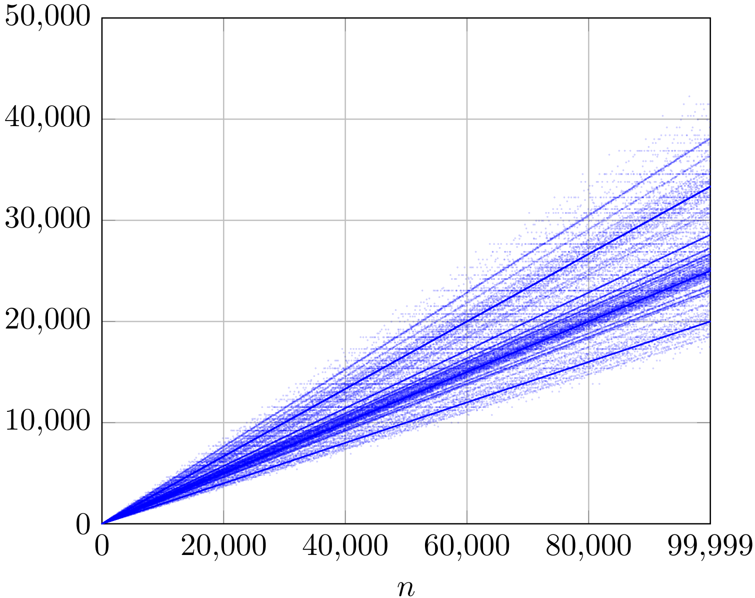
Non-symmetric Packings



$[-23, 48, 49, 52]$.

Non-symmetric Packings

Non-symmetric Packings



Families of non-symmetric packings

$n \equiv 0 \bmod 3 \implies$

Families of non-symmetric packings

$$n \equiv 0 \bmod 3 \implies \left[-n,\ n+9,\ \frac{n^2}{9}+n+1,\ \frac{n^2}{9}+n+4\right]$$

Families of non-symmetric packings

$$n \equiv 0 \bmod 3 \implies$$

$$\left[-n,\ n+9,\ \frac{n^2}{9}+n+1,\ \frac{n^2}{9}+n+4\right]$$

$$n \equiv 1 \bmod 5 \implies$$

Families of non-symmetric packings

$$n \equiv 0 \bmod 3 \implies \left[-n,\ n+9,\ \frac{n^2}{9}+n+1,\ \frac{n^2}{9}+n+4\right]$$

$$n \equiv 1 \bmod 5 \implies \left[-n,\ n+5,\ \frac{n^2+4}{5}+1,\ \frac{n^2+4}{5}+2\right]$$

Families of non-symmetric packings

$n \equiv 0 \bmod 3 \implies$

$$\left[-n,\ n+9,\ \frac{n^2}{9}+n+1,\ \frac{n^2}{9}+n+4\right]$$

$n \equiv 1 \bmod 5 \implies$

$$\left[-n,\ n+5,\ \frac{n^2+4}{5}+1,\ \frac{n^2+4}{5}+2\right]$$

Every packing can be written

Families of non-symmetric packings

$$n \equiv 0 \bmod 3 \implies$$

$$\left[-n,\ n+9,\ \frac{n^2}{9}+n+1,\ \frac{n^2}{9}+n+4\right]$$

$$n \equiv 1 \bmod 5 \implies$$

$$\left[-n,\ n+5,\ \frac{n^2+4}{5}+1,\ \frac{n^2+4}{5}+2\right]$$

Every packing can be written

$$\left[-n,\ n+k,\ \frac{n^2+kn+\alpha^2}{k},\ \frac{n^2+kn+(k-\alpha)^2}{k}\right]$$

(Bridges, Tai, and Koziol.)

Total Number of Packings

Total Number of Packings

The total packings of n is known:

Total Number of Packings

The total packings of n is known:

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+2^{\omega(n)-\delta_n-1},$$

Total Number of Packings

The total packings of n is known:

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+2^{\omega(n)-\delta_n-1},$$

where $\chi_{-4}(n)=(-1)^{(n-1)/2}$ for n odd and 0 for n even. (Due to Graham, Lagarias, Mallows, Wilks, Yan)

Total number of non-symmetric packings

Total number of non-symmetric packings

Corollary

Total number of non-symmetric packings

Corollary

The number of non-symmetric packings of n is given by

Total number of non-symmetric packings

Corollary

The number of non-symmetric packings of n is given by

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+\left(2^{\omega(n)-1}\right)\left(2^{-\delta_n}-2+\delta_n\right).$$

Total number of non-symmetric packings

Corollary

The number of non-symmetric packings of n is given by

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+\left(2^{\omega(n)-1}\right)\left(2^{-\delta_n}-2+\delta_n\right).$$

Proof.

Total number of non-symmetric packings

Corollary

The number of non-symmetric packings of n is given by

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+\left(2^{\omega(n)-1}\right)\left(2^{-\delta_n}-2+\delta_n\right).$$

Proof.

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+2^{\omega(n)-\delta_n-1}-\underbrace{(1-\delta_n)\cdot 2^{\omega(n)-1}}_{\text{twin-symmetric}}-\underbrace{2^{\omega(n)-1}}_{\text{sum-symmetric}}$$

Total number of non-symmetric packings

Corollary

The number of non-symmetric packings of n is given by

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+\left(2^{\omega(n)-1}\right)\left(2^{-\delta_n}-2+\delta_n\right).$$

Proof.

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+2^{\omega(n)-\delta_n-1}-\underbrace{(1-\delta_n)\cdot 2^{\omega(n)-1}}_{\text{twin-symmetric}}-\underbrace{2^{\omega(n)-1}}_{\text{sum-symmetric}}$$

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+\left(2^{\omega(n)-1}\right)\left(2^{-\delta_n}-2+\delta_n\right).$$

□

Extended Example

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$
Example: $20 = 2^2 \cdot 5$

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$
Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$
Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$

Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.

Total number is

$$\frac{20}{4} \prod_{p|20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20)-\delta_{20}-1}$$

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$
Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.
Total number is

$$\frac{20}{4} \prod_{p|20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20) - \delta_{20} - 1}$$

$$= 5\left(1 - \frac{\chi_{-4}(2)}{2}\right)\left(1 - \frac{\chi_{-4}(5)}{5}\right) + 2^{2-0-1}$$

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$
Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.
Total number is

$$\frac{20}{4} \prod_{p|20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20)-\delta_{20}-1}$$

$$= 5\left(1 - \frac{\chi_{-4}(2)}{2}\right)\left(1 - \frac{\chi_{-4}(5)}{5}\right) + 2^{2-0-1}$$

$$= 5\left(1 - \frac{0}{2}\right)\left(1 - \frac{1}{5}\right) + 2 = 5\left(\frac{4}{5}\right) + 2 = 6.$$

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$
Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.
Total number is

$$\frac{20}{4} \prod_{p|20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20)-\delta_{20}-1}$$

$$= 5\left(1 - \frac{\chi_{-4}(2)}{2}\right)\left(1 - \frac{\chi_{-4}(5)}{5}\right) + 2^{2-0-1}$$

$$= 5\left(1 - \frac{0}{2}\right)\left(1 - \frac{1}{5}\right) + 2 = 5\left(\frac{4}{5}\right) + 2 = 6.$$

- sum-symmetric: $2^{\omega(20)-1} = 2$.

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$
Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.
Total number is

$$\frac{20}{4} \prod_{p|20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20) - \delta_{20} - 1}$$

$$= 5 \left(1 - \frac{\chi_{-4}(2)}{2}\right) \left(1 - \frac{\chi_{-4}(5)}{5}\right) + 2^{2-0-1}$$

$$= 5 \left(1 - \frac{0}{2}\right) \left(1 - \frac{1}{5}\right) + 2 = 5 \left(\frac{4}{5}\right) + 2 = 6.$$

- sum-symmetric: $2^{\omega(20)-1} = 2$.
- twin-symmetric: $(1 - \delta_{20}) 2^{\omega(20)-1} = 2$.

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$
Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.
Total number is

$$\frac{20}{4} \prod_{p|20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20) - \delta_{20} - 1}$$

$$= 5\left(1 - \frac{\chi_{-4}(2)}{2}\right)\left(1 - \frac{\chi_{-4}(5)}{5}\right) + 2^{2-0-1}$$

$$= 5\left(1 - \frac{0}{2}\right)\left(1 - \frac{1}{5}\right) + 2 = 5\left(\frac{4}{5}\right) + 2 = 6.$$

- sum-symmetric: $2^{\omega(20)-1} = 2$.
- twin-symmetric: $(1 - \delta_{20})2^{\omega(20)-1} = 2$.

$\implies$ non-symmetric: $6 - 2 - 2 = 2$.

Extended Example

Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$

Example: $20 = 2^2 \cdot 5$, with $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$.

Total number is

$$\frac{20}{4} \prod_{p|20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20)-\delta_{20}-1}$$

$$= 5\left(1 - \frac{\chi_{-4}(2)}{2}\right)\left(1 - \frac{\chi_{-4}(5)}{5}\right) + 2^{2-0-1}$$

$$= 5\left(1 - \frac{0}{2}\right)\left(1 - \frac{1}{5}\right) + 2 = 5\left(\frac{4}{5}\right) + 2 = 6.$$

- sum-symmetric: $2^{\omega(20)-1} = 2$.
- twin-symmetric: $(1-\delta_{20})2^{\omega(20)-1} = 2$.

$\implies$ non-symmetric: $6 - 2 - 2 = 2$.

Coprime factor pairs of 20: $(1, 20)$ and $(4, 5)$.

Extended Example - sum-symmetric

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$$

$(1, 20)$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

$$(1, 20) \implies \left[-1 \cdot 20,\ 1(1+20),\ 20(1+20),\ (1+20)^2 - 1 \cdot 20\right]$$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$$

$$(1,20) \implies \left[-1\cdot 20,\ 1(1+20),\ 20(1+20),\ (1+20)^2-1\cdot 20\right]$$

$$= [-20,\ 21,\ 420,\ 421]$$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

$$(1,20) \implies \left[-1\cdot 20,\ 1(1+20),\ 20(1+20),\ (1+20)^2 - 1\cdot 20\right]$$

$$= [-20,\ 21,\ 420,\ 421]$$

$$(4,5)$$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

$$(1, 20) \implies \left[-1 \cdot 20,\ 1(1+20),\ 20(1+20),\ (1+20)^2 - 1 \cdot 20\right]$$

$$= [-20,\ 21,\ 420,\ 421]$$

$$(4, 5) \implies \left[-4 \cdot 5,\ 4(4+5),\ 5(4+5),\ (4+5)^2 - 4 \cdot 5\right]$$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

$$(1,20) \implies \left[-1\cdot 20,\ 1(1+20),\ 20(1+20),\ (1+20)^2 - 1\cdot 20\right]$$

$$= [-20,\ 21,\ 420,\ 421]$$

$$(4,5) \implies \left[-4\cdot 5,\ 4(4+5),\ 5(4+5),\ (4+5)^2 - 4\cdot 5\right]$$

$$= [-20,\ 36,\ 45,\ 61]$$

Extended Example - twin-symmetric

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \qquad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \qquad x > y \end{cases}$$

$(1, 10)$

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \qquad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \qquad x > y \end{cases}$$

$$(1, 10) \implies \left[-2\cdot 1\cdot 10,\ 2\cdot 1\cdot 10+4(1)^2,\ (1+10)^2,\ (1+10)^2\right]$$

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

$$(1, 10) \implies \left[-2\cdot 1\cdot 10,\ 2\cdot 1\cdot 10 + 4(1)^2,\ (1+10)^2,\ (1+10)^2\right]$$

$$= [-20,\ 24,\ 121,\ 121]$$

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

$$(1, 10) \implies \left[-2\cdot 1\cdot 10,\ 2\cdot 1\cdot 10 + 4(1)^2,\ (1+10)^2,\ (1+10)^2\right]$$

$$= [-20,\ 24,\ 121,\ 121]$$

$(2, 5)$

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x,y \text{ odd} \quad x>y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x>y \end{cases}$$

$$(1,10) \implies \left[-2\cdot 1\cdot 10,\ 2\cdot 1\cdot 10+4(1)^2,\ (1+10)^2,\ (1+10)^2\right]$$

$$= [-20,\ 24,\ 121,\ 121]$$

$$(2,5) \implies \left[-2\cdot 2\cdot 5,\ 2\cdot 2\cdot 5+4(2)^2,\ (2+5)^2,\ (2+5)^2\right]$$

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \qquad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \qquad x > y \end{cases}$$

$$(1, 10) \implies \left[-2 \cdot 1 \cdot 10,\ 2 \cdot 1 \cdot 10 + 4(1)^2,\ (1+10)^2,\ (1+10)^2\right]$$

$$= [-20,\ 24,\ 121,\ 121]$$

$$(2, 5) \implies \left[-2 \cdot 2 \cdot 5,\ 2 \cdot 2 \cdot 5 + 4(2)^2,\ (2+5)^2,\ (2+5)^2\right]$$

$$= [-20,\ 36,\ 49,\ 49]$$

Extended Example - non-symmetric

Extended Example - non-symmetric

$20 \equiv 7 \bmod 13 \implies$

Extended Example - non-symmetric

$$20 \equiv 7 \bmod 13 \implies$$

$$\left[-n,\ n+13,\ \left(\frac{n^2+13n+4^2}{13}\right),\ \left(\frac{n^2+13n+(13-4)^2}{13}\right)\right]$$

Extended Example - non-symmetric

$$20 \equiv 7 \bmod 13 \implies$$

$$\left[-n,\ n+13,\ \left(\frac{n^2+13n+4^2}{13}\right),\ \left(\frac{n^2+13n+(13-4)^2}{13}\right)\right]$$

$$= [-20,\ 33,\ 52,\ 57]$$

Extended Example - non-symmetric

$20 \equiv 7 \bmod 13 \implies$

$$\left[-n,\ n+13,\ \left(\frac{n^2+13n+4^2}{13}\right),\ \left(\frac{n^2+13n+(13-4)^2}{13}\right)\right]$$

$$= [-20,\ 33,\ 52,\ 57]$$

$20 \equiv 3 \bmod 17 \implies$

Extended Example - non-symmetric

$$20 \equiv 7 \bmod 13 \implies$$

$$\left[-n,\ n+13,\ \left(\frac{n^2+13n+4^2}{13}\right),\ \left(\frac{n^2+13n+(13-4)^2}{13}\right)\right]$$

$$= [-20,\ 33,\ 52,\ 57]$$

$$20 \equiv 3 \bmod 17 \implies$$

$$\left[-n,\ n+17,\ \left(\frac{n^2+17n+5^2}{17}\right),\ \left(\frac{n^2+17n+(17-5)^2}{17}\right)\right]$$

Extended Example - non-symmetric

$20 \equiv 7 \bmod 13 \implies$

$$\left[-n,\ n+13,\ \left(\frac{n^2+13n+4^2}{13}\right),\ \left(\frac{n^2+13n+(13-4)^2}{13}\right)\right]$$

$$= [-20,\ 33,\ 52,\ 57]$$

$20 \equiv 3 \bmod 17 \implies$

$$\left[-n,\ n+17,\ \left(\frac{n^2+17n+5^2}{17}\right),\ \left(\frac{n^2+17n+(17-5)^2}{17}\right)\right]$$

$$= [-20,\ 37,\ 45,\ 52]$$

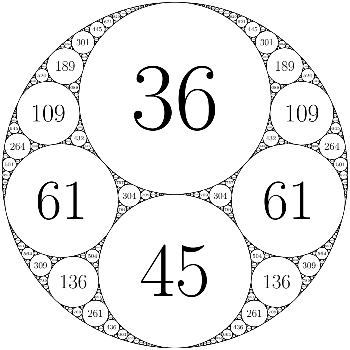
Extended Example

Extended Example

$[-20,\ 21,\ 420,\ 421]$

Extended Example

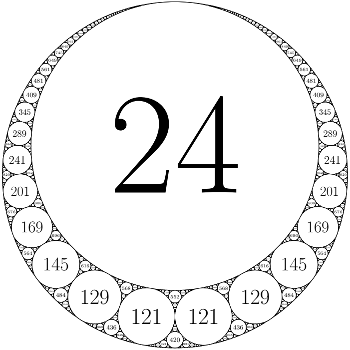
[−20, 21, 420, 421]



[−20, 36, 45, 61]

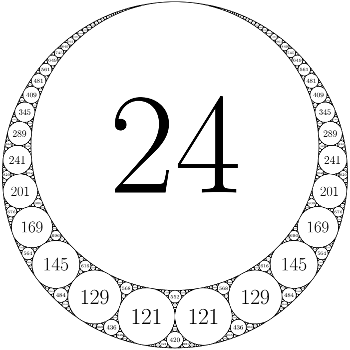
Extended Example

Extended Example

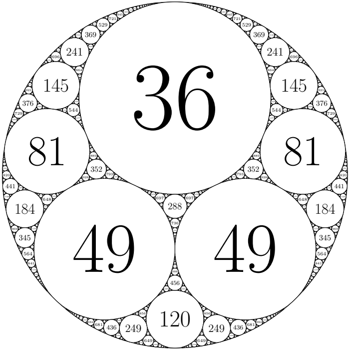


[−20, 24, 121, 121]

Extended Example



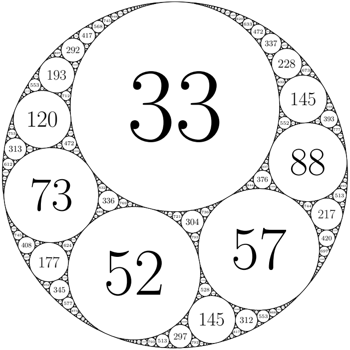
[−20, 24, 121, 121]



[−20, 36, 49, 49].

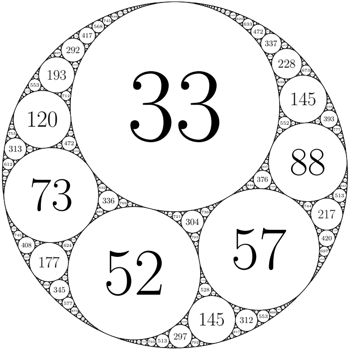
Extended Example

Extended Example

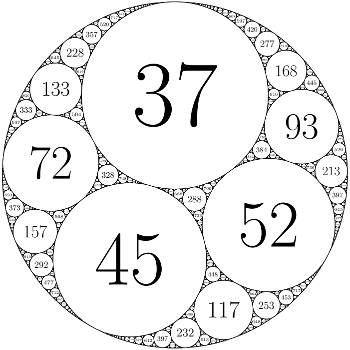


$[-20, 33, 52, 57]$.

Extended Example



[−20, 33, 52, 57].



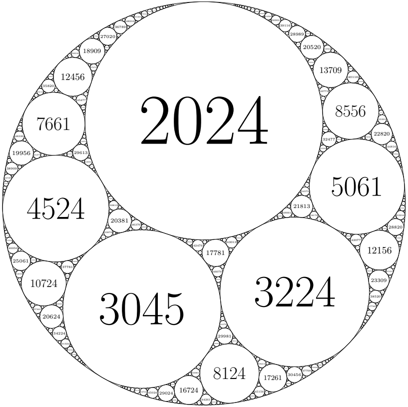
[−20, 37, 45, 52]

Acknowledgments

I am incredibly grateful to Professor Katherine Stange and Dr. James Rickards for taking me under their wing over the previous summer's REU. Their mentorship and encouragement inspired me to pursue not only this honors thesis, but a math conference across the country. Under their guidance, I have learned just how fun math research can be! Working on this thesis has been one of the most fulfilling projects I have undertaken.

I am also thankful to the Honors committee for reviewing my thesis. Without them, the honors program would not be possible.

Thank You!



Images generated using James Rickards' Code.

References

Tabes Bridges, Warren Tai, and Karol Koziol. A classification of integral apollonian circle packings. https://www.math.columbia.edu/-staff/RTGPapers/IntegralClassification.pdf, 2011.

Duncan Buell. Binary Quadratic Forms. Springer-Verlag, Berlin, 1989.

David A. Cox. Primes of the form $x^2 + ny^2$. Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, N.J, second edition, 2013. Fermat, class field theory, and complex multiplication.

Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian Circle Packings: Number Theory. J. Number Theory, 100(1):1-45, 2003.

James Rickards. Apollonian. https://github.com/JamesRickards-Canada/Apollonian, 2023.

Proof of Descartes Equation

Proof of Descartes Equation

First, we need a trigonometric lemma

Proof of Descartes Equation

First, we need a trigonometric lemma

Lemma

Proof of Descartes Equation

First, we need a trigonometric lemma

Lemma

If $\alpha + \beta + \theta = 2\pi$ *then*

Proof of Descartes Equation

First, we need a trigonometric lemma

Lemma

If $\alpha + \beta + \theta = 2\pi$ *then*

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \theta = 1 + 2\cos\alpha\cos\beta\cos\theta.$$

Proof of the Lemma

Proof of the Lemma

Proof.

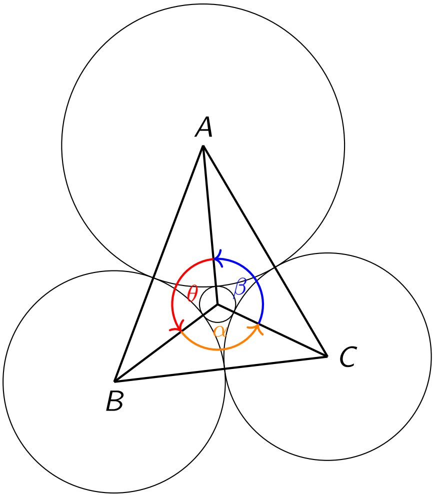
Proof of the Lemma

Proof.

$$
\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\theta &= \\
&= \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\theta}{2} \\
&= \frac{3}{2} + \frac{\cos 2\alpha + \cos 2\beta}{2} + \frac{\cos(2\pi - (2\alpha + 2\beta))}{2} \\
&= \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{\cos 2(\alpha+\beta)}{2} \\
&= \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{2\cos^2(\alpha+\beta) - 1}{2} \\
&= 1 + \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos^2(\alpha+\beta) \\
&= 1 + (\cos(\alpha-\beta) + \cos(\alpha+\beta))\cos(2\pi - \theta) \\
&= 1 + 2\cos\alpha\cos\beta\cos\theta.
\end{aligned}
$$

□

Proof of the Lemma



Four mutually tangent circles with centers A, B, C, and D.

Proof of the Descartes Equation

Proof of the Descartes Equation

Proof.

Proof.

Suppose we have four mutually tangent circles with centers A, B, C, and D

Proof of the Descartes Equation

Proof.

Suppose we have four mutually tangent circles with centers A, B, C, and D with respective radii r_A, r_B, r_C, and r_D.

Proof of the Descartes Equation

Proof.

Suppose we have four mutually tangent circles with centers A, B, C, and D with respective radii r_A, r_B, r_C, and r_D. The side lengths of $\triangle ABC$ are

$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad AC = r_A + r_C$$

Proof of the Descartes Equation

Proof.

Suppose we have four mutually tangent circles with centers A, B, C, and D with respective radii r_A, r_B, r_C, and r_D. The side lengths of $\triangle ABC$ are

$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad AC = r_A + r_C$$

and the lengths from the centers of circles A, B, C to D are

Proof of the Descartes Equation

Proof.

Suppose we have four mutually tangent circles with centers A, B, C, and D with respective radii r_A, r_B, r_C, and r_D. The side lengths of $\triangle ABC$ are

$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad AC = r_A + r_C$$

and the lengths from the centers of circles A, B, C to D are

$$AD = r_A + r_D, \quad BD = r_B + r_D, \quad CD = r_C + r_D.$$

Proof of the Descartes Equation

Proof.

Suppose we have four mutually tangent circles with centers A, B, C, and D with respective radii r_A, r_B, r_C, and r_D. The side lengths of $\triangle ABC$ are

$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad AC = r_A + r_C$$

and the lengths from the centers of circles A, B, C to D are

$$AD = r_A + r_D, \quad BD = r_B + r_D, \quad CD = r_C + r_D.$$

Let $\angle BDC = \alpha$, $\angle CDA = \beta$, and $\angle ADB = \theta$.

Proof of the Descartes Equation

Proof.

Suppose we have four mutually tangent circles with centers A, B, C, and D with respective radii r_A, r_B, r_C, and r_D. The side lengths of $\triangle ABC$ are

$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad AC = r_A + r_C$$

and the lengths from the centers of circles A, B, C to D are

$$AD = r_A + r_D, \quad BD = r_B + r_D, \quad CD = r_C + r_D.$$

Let $\angle BDC = \alpha$, $\angle CDA = \beta$, and $\angle ADB = \theta$. The law of cosines in $\triangle ADB$ yields

$$\begin{aligned}
\cos\theta &= \frac{AD^2 + BD^2 - AB^2}{2 \cdot AD \cdot BD} \\
&= \frac{(r_A + r_D)^2 + (r_B + r_D)^2 - (r_A + r_B)^2}{2(r_A + r_D)(r_B + r_D)} \\
&= \frac{2r_D^2 + 2r_D(r_A + r_B) - 2r_A r_B}{2(r_A + r_D)(r_B + r_D)} \\
&= 1 - \frac{2r_A r_B}{(r_A + r_D)(r_B + r_D)}.
\end{aligned}$$

Proof of the Descartes Equation

Proof.

Proof.

Similarly, we find in $\triangle ADB$ and $\triangle CDA$ that

Proof.

Similarly, we find in $\triangle ADB$ and $\triangle CDA$ that

$$\cos\alpha = 1 - \frac{2r_B r_C}{(r_B + r_D)(r_C + r_D)}, \quad \cos\beta = 1 - \frac{2r_A r_C}{(r_A + r_D)(r_C + r_D)}.$$

Proof.

Similarly, we find in $\triangle ADB$ and $\triangle CDA$ that

$$\cos\alpha = 1 - \frac{2r_B r_C}{(r_B + r_D)(r_C + r_D)}, \quad \cos\beta = 1 - \frac{2r_A r_C}{(r_A + r_D)(r_C + r_D)}.$$

Now replace each radius by it's respective curvature k_A, k_B, k_C, and k_D and name the associated fraction to each angle λ

Proof.

Similarly, we find in $\triangle ADB$ and $\triangle CDA$ that

$$\cos\alpha = 1 - \frac{2r_B r_C}{(r_B + r_D)(r_C + r_D)}, \quad \cos\beta = 1 - \frac{2r_A r_C}{(r_A + r_D)(r_C + r_D)}.$$

Now replace each radius by it's respective curvature k_A, k_B, k_C, and k_D and name the associated fraction to each angle λ

$$\cos\alpha = 1 - \frac{2k_D^2}{(k_B + k_D)(k_C + k_D)} = 1 - \lambda_\alpha$$

$$\cos\beta = 1 - \frac{2k_D^2}{(k_A + k_D)(k_C + k_D)} = 1 - \lambda_\beta$$

$$\cos\theta = 1 - \frac{2k_D^2}{(k_A + k_D)(k_B + k_D)} = 1 - \lambda_\theta.$$

Proof of the Descartes Equation

Proof.

Proof of the Descartes Equation

Proof.

By the lemma we have that

Proof of the Descartes Equation

Proof.

By the lemma we have that

$$(1-\lambda_\alpha)^2 + (1-\lambda_\beta)^2 + (1-\lambda_\theta)^2 = 1 + 2(1-\lambda_\alpha)(1-\lambda_\beta)(1-\lambda_\theta)$$
$$\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\theta^2 + 2\lambda_\alpha\lambda_\beta\lambda_\theta = 2(\lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\theta + \lambda_\alpha\lambda_\theta)$$
$$\frac{\lambda_\alpha}{\lambda_\beta\lambda_\theta} + \frac{\lambda_\beta}{\lambda_\alpha\lambda_\theta} + \frac{\lambda_\theta}{\lambda_\alpha\lambda_\beta} + 2 = 2\left(\frac{1}{\lambda_\alpha} + \frac{1}{\lambda_\beta} + \frac{1}{\lambda_\theta}\right).$$

Proof of the Descartes Equation

Proof.

By the lemma we have that

$$(1-\lambda_\alpha)^2 + (1-\lambda_\beta)^2 + (1-\lambda_\theta)^2 = 1 + 2(1-\lambda_\alpha)(1-\lambda_\beta)(1-\lambda_\theta)$$

$$\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\theta^2 + 2\lambda_\alpha\lambda_\beta\lambda_\theta = 2(\lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\theta + \lambda_\alpha\lambda_\theta)$$

$$\frac{\lambda_\alpha}{\lambda_\beta\lambda_\theta} + \frac{\lambda_\beta}{\lambda_\alpha\lambda_\theta} + \frac{\lambda_\theta}{\lambda_\alpha\lambda_\beta} + 2 = 2\left(\frac{1}{\lambda_\alpha} + \frac{1}{\lambda_\beta} + \frac{1}{\lambda_\theta}\right).$$

Substituting back our values for the λs we find

Proof.

By the lemma we have that

$$(1-\lambda_\alpha)^2 + (1-\lambda_\beta)^2 + (1-\lambda_\theta)^2 = 1 + 2(1-\lambda_\alpha)(1-\lambda_\beta)(1-\lambda_\theta)$$

$$\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\theta^2 + 2\lambda_\alpha\lambda_\beta\lambda_\theta = 2(\lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\theta + \lambda_\alpha\lambda_\theta)$$

$$\frac{\lambda_\alpha}{\lambda_\beta\lambda_\theta} + \frac{\lambda_\beta}{\lambda_\alpha\lambda_\theta} + \frac{\lambda_\theta}{\lambda_\alpha\lambda_\beta} + 2 = 2\left(\frac{1}{\lambda_\alpha} + \frac{1}{\lambda_\beta} + \frac{1}{\lambda_\theta}\right).$$

Substituting back our values for the λs we find

$$\frac{(k_A+k_D)^2}{2k_D^2} + \frac{(k_B+k_D)^2}{2k_D^2} + \frac{(k_C+k_D)^2}{2k_D^2} + 2 =$$

$$2\frac{(k_B+k_D)(k_C+k_D)}{2k_D^2} + 2\frac{(k_A+k_D)(k_B+k_D)}{2k_D^2}$$

$$+2\frac{(k_A+k_D)(k_B+k_D)}{2k_D^2}.$$

Proof of the Descartes Equation

Proof.

Proof.

We multiply through by $2k_d^2$ and simplfy to find that

Proof of the Descartes Equation

Proof.

We multiply through by $2k_d^2$ and simplfy to find that

$$\begin{aligned} k_A^2 + k_B^2 + k_C^2 + 2k_D(k_A + k_B + k_C) + 7k_D^2 \\ = 6k_D^2 + 4k_D(k_A + k_B + k_C) \\ + 2(k_A k_B + k_B k_C + k_A k_C) \end{aligned}$$

Proof of the Descartes Equation

Proof.

We multiply through by $2k_d^2$ and simplfy to find that

$$\begin{aligned} k_A^2 + k_B^2 + k_C^2 + 2k_D(k_A + k_B + k_C) + 7k_D^2 \\ = 6k_D^2 + 4k_D(k_A + k_B + k_C) \\ + 2(k_Ak_B + k_Bk_C + k_Ak_C) \end{aligned}$$

$$\begin{aligned} k_A^2 + k_B^2 + k_C^2 + k_D^2 &= 2k_D(k_A + k_B + k_C) \\ &\quad + 2(k_Ak_B + k_Bk_C + k_Ak_C) \\ &= (k_A + k_B + k_C + k_D)^2 \\ &\quad - (k_A^2 + k_B^2 + k_C^2 + k_D^2) \\ 2(k_A^2 + k_B^2 + k_C^2 + k_D^2) &= (k_A + k_B + k_C + k_D)^2. \end{aligned}$$

□

Proving the Parameterization

Clyde Kertzer

Proving the Parameterization

Proposition

Proving the Parameterization

Proposition

The following equalities hold in a sum-symmetric packing $[a, b, c, d]$*.*

Proving the Parameterization

Proposition

The following equalities hold in a sum-symmetric packing $[a, b, c, d]$.

(i) $a + b = d - c$

(ii) $d^2 = a^2 + b^2 + c^2$

(iii) $ab + ac + bc = 0$

Proving the Parameterization

Proposition

The following equalities hold in a sum-symmetric packing $[a, b, c, d]$.

(i) $a + b = d - c$

(ii) $d^2 = a^2 + b^2 + c^2$

(iii) $ab + ac + bc = 0$

(i) We know that a sum-symmetric packing has the property that

$$2(a + b + c) - d = d.$$

Proving the Parameterization

Proposition

The following equalities hold in a sum-symmetric packing $[a, b, c, d]$*.*

(i) $a + b = d - c$

(ii) $d^2 = a^2 + b^2 + c^2$

(iii) $ab + ac + bc = 0$

(i) We know that a sum-symmetric packing has the property that

$$2(a + b + c) - d = d.$$

This yields immediately

Proving the Parameterization

Proposition

The following equalities hold in a sum-symmetric packing $[a, b, c, d]$.

(i) $a + b = d - c$

(ii) $d^2 = a^2 + b^2 + c^2$

(iii) $ab + ac + bc = 0$

(i) We know that a sum-symmetric packing has the property that

$$2(a + b + c) - d = d.$$

This yields immediately

$$a + b + c = d$$
$$a + b = d - c.$$

Proving the Parameterization

Proving the Parameterization

(ii) Plugging part (i) back into the Descartes Equation we find

Proving the Parameterization

(ii) Plugging part (i) back into the Descartes Equation we find

$$\begin{aligned}(a+b+c+d)^2 &= 2\left(a^2+b^2+c^2+d^2\right)\\(d-c+c+d)^2 &= 2a^2+2b^2+2c^2+2d^2\\4d^2 &= 2\left(a^2+b^2+c^2\right)+2d^2\\d^2 &= a^2+b^2+c^2.\end{aligned}$$

Proving the Parameterization

(ii) Plugging part (i) back into the Descartes Equation we find

$$(a+b+c+d)^2 = 2\left(a^2+b^2+c^2+d^2\right)$$
$$(d-c+c+d)^2 = 2a^2+2b^2+2c^2+2d^2$$
$$4d^2 = 2\left(a^2+b^2+c^2\right)+2d^2$$
$$d^2 = a^2+b^2+c^2.$$

(iii) Use substitutions from parts (i) and (ii) to find

Proving the Parameterization

(ii) Plugging part (i) back into the Descartes Equation we find

$$\begin{aligned}(a+b+c+d)^2 &= 2\left(a^2+b^2+c^2+d^2\right)\\(d-c+c+d)^2 &= 2a^2+2b^2+2c^2+2d^2\\4d^2 &= 2\left(a^2+b^2+c^2\right)+2d^2\\d^2 &= a^2+b^2+c^2.\end{aligned}$$

(iii) Use substitutions from parts (i) and (ii) to find

$$\begin{aligned}a+b+c &= d\\(a+b+c)^2 &= a^2+b^2+c^2\\a^2+b^2+c^2+2ab+2ac+2bc &= a^2+b^2+c^2\\ab+ac+bc &= 0\end{aligned}$$

Proving the Parameterization

Proving the Parameterization

Proof.

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$.

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

$$ab + ac + bc = 0$$
$$a^2 + ab + ac + bc = a^2$$
$$(a + b)(a + c) = a^2.$$

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

$$ab + ac + bc = 0$$

$$a^2 + ab + ac + bc = a^2$$

$$(a + b)(a + c) = a^2.$$

Let $g = \gcd(a + b, a + c)$

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

$$ab + ac + bc = 0$$
$$a^2 + ab + ac + bc = a^2$$
$$(a + b)(a + c) = a^2.$$

Let $g = \gcd(a + b, a + c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y.

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

$$ab + ac + bc = 0$$
$$a^2 + ab + ac + bc = a^2$$
$$(a + b)(a + c) = a^2.$$

Let $g = \gcd(a + b, a + c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y. This yields $gxy = -a$.

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

$$ab + ac + bc = 0$$
$$a^2 + ab + ac + bc = a^2$$
$$(a + b)(a + c) = a^2.$$

Let $g = \gcd(a + b, a + c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y. This yields $gxy = -a$. Now, we have

$$b = (a + b) + (-a) = gx^2 + gxy$$

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

$$ab + ac + bc = 0$$

$$a^2 + ab + ac + bc = a^2$$

$$(a+b)(a+c) = a^2.$$

Let $g = \gcd(a+b, a+c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y. This yields $gxy = -a$. Now, we have

$$b = (a+b) + (-a) = gx^2 + gxy$$

and

$$c = (a+c) + (-a) = gy^2 + gxy.$$

Proving the Parameterization

Proof.

Proof.

Using the relation $d = a + b + c$ we can substitute what we have just found to find

Proving the Parameterization

Proof.

Using the relation $d = a + b + c$ we can substitute what we have just found to find

$$\begin{aligned} d &= a + b + c \\ &= (-gxy) + (gx^2 + gxy) + (gy^2 + gxy) \\ &= g((x + y)^2 - xy). \end{aligned}$$

Proving the Parameterization

Proof.

Using the relation $d = a + b + c$ we can substitute what we have just found to find

$$\begin{aligned} d &= a + b + c \\ &= (-gxy) + (gx^2 + gxy) + (gy^2 + gxy) \\ &= g((x + y)^2 - xy). \end{aligned}$$

Thus, we have that

$$\begin{aligned} a &= -gxy \\ b &= gx(x + y) \\ c &= gy(x + y) \\ d &= g((x + y)^2 - xy). \end{aligned}$$

Proving the Parameterization

Proof.

Proving the Parameterization

Proof.

Clearly, for the quadruple to be primitive, g must be 1, meaning x and y are coprime.

Proving the Parameterization

Proof.

Clearly, for the quadruple to be primitive, g must be 1, meaning x and y are coprime. Thus, we have

$$\begin{aligned} a &= -xy \\ b &= x(x+y) \\ c &= y(x+y) \\ d &= (x+y)^2 - xy. \end{aligned}$$

with $\gcd(x, y) = 1$. □