PARAMETERIZATIONS OF DESCARTES QUADRUPLES

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ABSTRACT. A Descartes quadruple is a set of four mutually tangent circles. A Descartes quadruples generates what is known as an Apollonian circle packing. By studying the ratios of the circles' curvatures, two distinct types of symmetric packings quadruples appear. We give parameterizations of these Descartes quadruples and count how many packings of each type are contained by a given enclosing circle.

CONTENTS

1. Introduction: Upper Half Plane & Möbius Transformations

We define the *projective space of dimension n over field $\mathbb{F}$*, written $\mathbb{P}^n_{\mathbb{F}}$, as $\frac{\mathbb{F}^{n+1}\backslash\{0\}}{\sim}$, where we have an equivalence relation $\sim$ between two vectors $v_1 \sim v_2$ if $v_1 = \lambda v_2$ for some $\lambda \in \mathbb{F}^*$. Often, a vector is scaled so that the last coordinate is 1. For example, $[1,7] \sim [2,14] \sim [1/7,1]$. This means that $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^2/\sim = \mathbb{C} \cup \{\infty\}$. In other words

$$[z,w] \sim \begin{cases} [1,0] & \text{if } w = 0 \\ [z/w,1] & \text{if } w \neq 0. \end{cases}$$

Now define linear maps on $M_2(\mathbb{C})$ so invertible linear maps are $\mathrm{GL}_2(\mathbb{C})$. Thus, we have that $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})\backslash\{\lambda I : \lambda \in \mathbb{C}^*\}$. These linear fraction transformations $\mathrm{PSL}_2(\mathbb{C})$ are called *Möbius transformations.* Explicitly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}) \text{ acts on } \begin{pmatrix} z \\ w \end{pmatrix} \text{ via } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}.$$

On $\mathbb{C} \cup \{\infty\}$ and for $w \neq 0$ we have

$$\frac{z}{w} \in \mathbb{C} \longmapsto \frac{az+bw}{cz+dw} = \frac{a\frac{z}{w}+b}{c\frac{z}{w}+d} \in \mathbb{C}$$

which means

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a}{c}.$$

It is straightforward to check that this is a group action where composition is matrix multiplication. Note that matrix multiplication is *not* commutative. Moreover, Möbius transformations are *conformal*, meaning they preserve angles between circles and they map circles to circles. We understand a straight line as a circle through infinity. The three special types of Möbius transformations are

1. Translations: $z \longmapsto z+a$: $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$
2. Scaling/rotation $z \longmapsto az$: $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{pmatrix}$
3. Inversion: $z \longmapsto \dfrac{-1}{z}$: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The *upper half plane model* is given by $\mathcal{H} = \{x+iy \in \mathbb{C} : y > 0\}$. Here the boundary is $\mathbb{R} \cup \{\infty\}$. We define the *modular group* $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ which is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ that acts on $\mathcal{H}$. It is generated by the two elements $S : z \to -1/z$ and $T : z \to z+1$. The modular group has the presentation

$$\Gamma \cong \left\langle S, T : S^2 = 1, (ST)^3 = 1 \right\rangle.$$

Note S and T have matrix representations

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These structures will be useful in our study of quadratic forms.

2. Quadratic Forms

A *binary quadratic form* is a function $Q(x,y) = Ax^2 + Bxy + Cy^2$ for some $A, B, C \in \mathbb{Z}$. For simplicity, we will refer to binary quadratic forms as *forms*. A form is *primitive* if $\gcd(A,B,C) = 1$. We often write $Q(x,y) = Ax^2 + Bxy + Cy^2$ as $Q = [A,B,C]$. A form *represents* a number n if there exists integers x and y such that $Q(x,y) = n$. If x and y are coprime, we say n is *properly represented*.

Example 2.1. We state a familiar result of elementary number theory using forms. The form $Q(x,y) = x^2 + y^2 = [1,0,1]$ properly represents an odd prime p if and only if $p \equiv 1 \pmod 4$.

The *discriminant* D of $[A,B,C]$ is defined by $D = B^2 - 4AC$. If $D < 0$, Q is *definite*. If $D = 0$, then Q is *degenerate* and Q will factor as $Q = (ax+by)^2$. If $D > 0$ and nonsquare, Q is *indefinite*. An indefinite form will represent both positive and negative numbers. We say Q is *positive definite* if Q only properly represents positive numbers and *negative definite* if Q only properly represents negative numbers. If we know the sign of both D and A, we can tell if a form is positive or negative definite, or indefinite. This yields the following proposition.

Proposition 2.2. *A form Q is positive definite if and only if $D < 0$ and $A > 0$.*

Proof. We complete the square to find

$$\begin{aligned} Q = Ax^2 + Bxy + Cy^2 &= Ax^2 + Bxy + \frac{B^2y^2}{4A} + Cy^2 - \frac{B^2y^2}{4A} \\ &= A\left(x^2 + \frac{B}{A}xy + \frac{B^2y^2}{4A^2}\right) + y^2\left(C - \frac{B^2}{4A}\right) \\ &= A\left(x + \frac{B}{2A}y\right)^2 + y^2\left(\frac{-D}{4A}\right). \end{aligned}$$

If Q is positive definite, then since $Q(1,0) = A$, A must be positive. Since $Q(-\frac{b}{2a}, 1) = \frac{-D}{4a}$, $D < 0$. The converse is clear. If $D < 0$ and $A > 0$, then

$$A\left(x + \frac{B}{2A}y\right)^2 + y^2\left(\frac{-D}{4A}\right) > 0. \qquad \square$$

This same proof shows that a form Q is negative definite if and only if $D < 0$ and $A < 0$. The proof is the exact same as that of positive definite. Also, note that $D = B^2 - 4AC \equiv B^2 \mod 4 \equiv 0, 1$.

Example 2.3. Let $Q(x,y) = 10x^2 + 14xy + 5y^2$ with $D = 14^2 - 4 \cdot 10 \cdot 5 = -4 < 0$. We will find the numbers it properly represents. We can write Q as a sum of squares as $Q = (3x+2y)^2 + (x+y)^2$. Now change variables by letting $x' = 3x + 2y$ and $y' = x + y$ so $Q' = (x')^2 + (y')^2 = [1, 0, 1]$. Writing in matrix form we see this is a Möbius transformation.

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad \text{and note that} \quad \det\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = 3 \cdot 1 - 2 \cdot 1 = 1.$$

Proposition 2.4. *The action of* $\mathrm{PSL}_2(\mathbb{Z})$ *on a quadratic form by* $g \cdot Q = Q(ax + by, cx + dy)$

(1) is a right group action,
(2) preserves the discriminant of Q, and
(3) preserves the set of properly represented numbers.

Proof. Take

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}).$$

We will first show that $g \circ (h \circ Q) = (gh) \circ Q$, that is, the action of $\mathrm{PSL}_2(\mathbb{Z})$ on quadratic forms is a right action. We have

$$\begin{aligned} g \circ (h \circ Q) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} \circ Q(x,y)\right] \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ Q(ex + fy, gx + hy) \\ &= Q\left(a(ex + fy) + b(gx + hy),\ c(ex + fy) + d(gx + hy)\right) \\ &= Q\left(x(ae + bg) + y(af + bh),\ x(ce + dg) + y(cf + dh)\right) \\ &= \begin{pmatrix} ae + gb & af + bh \\ ce + dg & cf + dh \end{pmatrix} \circ Q(x,y) \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] \circ Q(x,y) = (gh) \circ Q(x,y) \end{aligned}$$

and we have proved (1).

To see that the action preserves discriminant, observe that

$$\begin{aligned}
Q(ax+by, cx+dy) &= A(ax+by)^2 + B(ax+by)(cx+dy) + C(cx+dy)^2 \\
&= A(a^2x^2 + 2abxy + b^2y^2) + B(acx^2 + adxy + bcyx + bdy^2) \\
&\quad + C(c^2x^2 + 2cdxy + d^2y^2) \\
&= x^2\left(Aa^2 + Bac + Cc^2\right) + xy(2Aab + Bad + Bbc + 2Ccd) \\
&\quad + y^2\left(Ab^2 + Bbd + Cd^2\right)
\end{aligned}$$

so

$$\begin{aligned}
D(Q) &= (2Aab + Bad + Bbc + 2Ccd)^2 - 4\left(Aa^2 + Bac + Cc^2\right)\left(Ab^2 + Bbd + Cd^2\right) \\
&= 4A^2a^2b^2 + 2ABa^2b^2 + 2ABAb^2c + 4ACabcd + 2ABa^2bd + B^2a^2b^2 + B^2ab^2c \\
&\quad + 2BCacd^2 + 2ABab^2c + B^2abcd + B^2b^2c^2 + 2BCbc^2d + 4ACabcd \\
&\quad + 2BCacd^2 + 2Bbc^2d + 4C^2bc^2d - 4\left(A^2a^2b^2v + ABa^2bd + ACa^2d^2\right. \\
&\quad \left. + ABab^2C + B^2abcd + BCacd^2 + \ Ab^2c^2 + BCbc^2d + C^2c^2d^2\right) \\
&= 8ACabcd + B^2a^2d^2 - 2B^2abcd + B^2b^2c^2 - 4ACa^2d^2 - 4ACb^2c^2 \\
&= a^2d^2\left(B^2 - 4AC\right) + b^2c^2\left(B^2 - 4AC\right) + 2abcd\left(B^2 - 4AC\right) \\
&= \left(B^2 - 4AC\right)(ad - bc)^2 = B^2 - 4AC
\end{aligned}$$

where we have used $ad - bc = 1$ and we have proved (2). More generally, we have showed that the discriminant scales by the square of $ad - bc$.

To see that the transformation preserves the set of properly represented numbers, we need to show that $g \cdot Q = Q(ax+by, cx+dy)$ holds when $g = S$ and $g = T$. We have

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{aligned}
S \cdot Q(x,y) &= Q((0)x + (-1)y, (1)x + (0)y) \\
&= A(-y)^2 + B(-y)x + Cx^2 \\
&= Ay^2 - Bxy + Cx^2
\end{aligned}$$

and

$$\begin{aligned}
T \cdot Q(x,y) &= Q((1)x + (1)y, (0)x + (1)y) \\
&= A(x+y)^2 + B(x+y)x + Cy^2 \\
&= A(x^2 + 2xy + y^2) + B(x^2 + xy) + Cy^2 \\
&= Ax^2 + 2Axy + Ay^2 + Bx^2 + Bxy + Cy^2 \\
&= (A+B)x^2 + (2A+B)xy + (A+C)y^2.
\end{aligned}$$

□

For the purposes of circle packing, we will focus on forms with $D < 0$, so from now on we will assume $D < 0$. These forms also have a notion of simplicity. Two forms may be different yet represent the exact same numbers. Take the forms $Q = x^2 + xy + y^2$ and $Q' = 3x^2 + 3xy + y^2$ both with $D = -3$. If we write out the first few numbers they represent we find that

$$Q \text{ represents } 1, 2, 3, 7, 13, \ldots \quad \text{and} \quad Q' \text{ represents } 1, 2, 3, 7, 13, \ldots$$

In fact, these two forms represent the exact same numbers. This is because Q' *reduces* to Q. A positive definite form $[A, B, C]$ is *reduced* if $|B| \leqslant A \leqslant C$ and if either equality holds, $B \geqslant 0$. Two forms are *equivalent* if their reduced forms are the same (up to the parity of B). So, Q and Q' are in fact equivalent. Every form is equivalent to a unique reduced form. In general, it is better to work with reduced forms, and we have the following reduction algorithm for negative discriminants:

1. Apply T^n so that $2|B| \leqslant A$.
2. If $A > C$, apply S to swap A and C.
3. Terminate when $|B| \leqslant A \leqslant C$. If $|B| = A$, $B \geqslant 0$ and if $A = C$, $B \geqslant 0$.

Example 2.5. Let $Q = [7, 13, 11]$ with $D = 169 - 4(7)(11) = -139$. Then we have that

$$\begin{aligned} T^{-1}[7, 13, 11] &= [7, -1, 5] \\ ST^{-1}[7, 13, 11] &= [5, 1, 7]. \end{aligned}$$

If $Q = [31, 16, 3]$ we have

$$\begin{aligned} S[31, 16, 3] &= [3, -16, 31] \\ T[3, -16, 3] &= [3, -10, 18] \\ T[3, -10, 18] &= [3, -4, 11] \\ T[3, -4, 11] &= [3, 2, 10]. \end{aligned}$$

Proposition 2.6. *There are finitely many reduced forms with negative discriminant D.*

Proof. Using the fact that a reduced form satisfies $|B| \leqslant A \leqslant C$ we have

$$\begin{aligned} D &= B^2 - 4AC \\ &\leqslant A^2 - 4AC \\ &\leqslant A^2 - 4A^2 = -3A^2 \end{aligned}$$

so

$$\begin{aligned} 3A^2 &\leqslant -D \\ 0 < A &\leqslant \sqrt{-D/3} \end{aligned}$$

so there are finitely many choices for A. Given A, there are finitely many choices for B and thus C is determined uniquely. Therefore, there are finitely many reduced forms with discriminant D. $\square$

We define the *narrow class group* $\mathrm{cl}(D) = \{\text{equivalence classes of forms of disc } D\}$ which has a group law where two forms are equivalent if their reduced forms are equal. This yields the *narrow class number* $\mathrm{h}(D) = |\,\mathrm{cl}(D)|$, the number of reduced forms with discriminant D. We have the rough estimate $\mathrm{h}(D) \approx \sqrt{-D}$.

Example 2.7. Let $D = -4$ with $\mathrm{h}(D) = 1$ so $Q(x, y) = x^2 + y^2$. Let p be a prime $p \equiv 1 \pmod 4$. Then there is a solution to $B^2 = -1 \pmod p$:

$$\begin{aligned} p &\mid B^2 + 1 \\ 4p &\mid (2B)^2 + 4 \\ 4pC &= (2B)^2 + 4 \qquad \text{for some } C \in \mathbb{Z} \\ (2B)^2 - 4pC &= -4 \end{aligned}$$

so $[p, 2B, C]$ has $D = -4$ and thus we have shown that p is a sum of two squares.

Example 2.8. We will use reduction to show that $Q(x, y) = [2, 1, 10]$ with $D = -79$ and $R(x, y) = [4, -1, 5]$ also with $D = -79$ are not equivalent. We find that Q represents 2 at $(1, 0)$ but that R cannot represent 2 as $4x^2 + xy + 5y^2 \geqslant 3x^2 + 5y^2 > 2$ for $x, y \geqslant 1$.

Example 2.9. Let $D = -23$. Then we have that

$$\begin{aligned} -23 &= B^2 - 4AC \\ C &= \frac{B^2 + 23}{4A}. \end{aligned}$$

We require $|B| \leqslant A \leqslant C$. Clearly B must be odd and

$$\begin{aligned} 0 < A &< \sqrt{23/3} \\ 1 \leqslant A &\leqslant 2 \end{aligned}$$

so we test the cases $A = 1, 2$. When $A = 2$, $C = (B^2 + 23)/8$ so $B = \pm 1$, giving the forms $[2, \pm 1, 3]$. When $A = 1$, $C = (B^2 + 23)/4$ so $B = \pm 1$, giving the forms $[1, \pm 1, 3]$. However, we exclude the equivalent case $[1, -1, 3]$ so $\mathrm{h}^+(-23) = 3$.

3. Apollonian Circle Packings

We are now prepared for our study of Apollonian circle packings, the main topic of this thesis. Let's begin by defining a *Descartes quadruple* as a set of four mutually tangent circles with disjoint interiors, written (A, B, C, D). Apollonius of Perga, who first studied Apollonian circle packings (hence the name) found that if three circles are mutually tangent, there are exactly two circles that are tangent to all three.

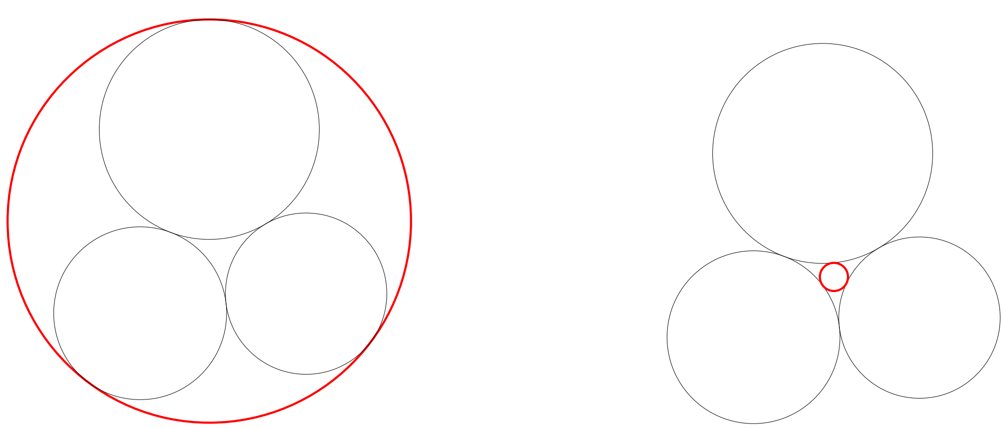


Figure 1. A demonstration of the Theorem of Apollonius.

Remark. Figure 1 and all upcoming figures were generated using the code of [Ric23].

The circles in the second quadruple of Figure 1 clearly have disjoint interiors, so it is a Descartes quadruple. However, we wish for the first quadruple to also be a Descartes quadruple. So, we can view the enclosing circle as an "inverted" circle, so its interior is everything outside of it. We can have at most one inverted circle in a Descartes quadruple, as its interior is infinite.

Now, given a circle with radius r, we define the circle's *curvature* to be $1/r$. This definition allows us to interpret a line as a circle with infinite radius having curvature 0. Additionally, the curvature of the enclosing inverted circle is given a negative sign. For sake of simplicity we will also refer to a quadruple of four mutually tangent circles' curvatures as a Descartes quadruple, written $[a, b, c, d]$. This definition yields a fundamental relation between the curvatures of a Descartes quadruple due to Descartes.

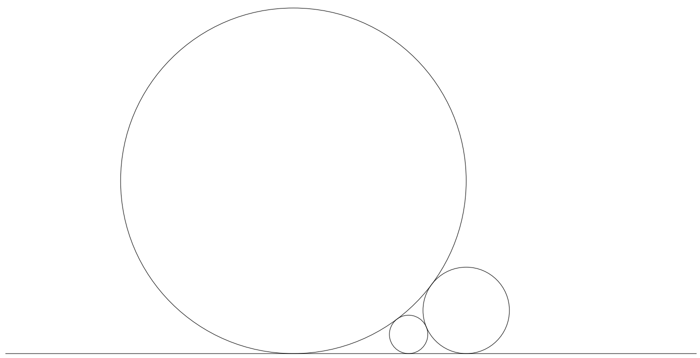


FIGURE 2. A Descartes quadruple with a circle of infinite radius: a straight line!

Theorem 3.1 (Descartes' Equation)**.** *If four circles in a Descartes quadruple have respective curvatures k_A, k_B, k_C, and k_D then*

$$2\left(k_A^2 + k_B^2 + k_C^2 + k_D^2\right) = \left(k_A + k_B + k_C + k_D\right)^2.$$

To prove Descartes' equation, we need a trigonometric lemma.

Lemma 3.2. *If $\alpha + \beta + \theta = 2\pi$ then*

$$\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1 + 2\cos\alpha\cos\beta\cos\theta.$$

Proof. Suppose that $\alpha + \beta + \theta = 2\pi$. Then using standard trigonometric identities we have

$$\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\theta &= \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\theta}{2} \\
&= \frac{3}{2} + \frac{\cos 2\alpha + \cos 2\beta}{2} + \frac{\cos(2\pi - (2\alpha + 2\beta))}{2} \\
&= \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{\cos 2(\alpha+\beta)}{2} \\
&= \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{2\cos^2(\alpha+\beta) - 1}{2} \\
&= 1 + \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos^2(\alpha+\beta) \\
&= 1 + (\cos(\alpha-\beta) + \cos(\alpha+\beta))\cos(2\pi - \theta) \\
&= 1 + 2\cos\alpha\cos\beta\cos\theta.
\end{aligned}$$

□

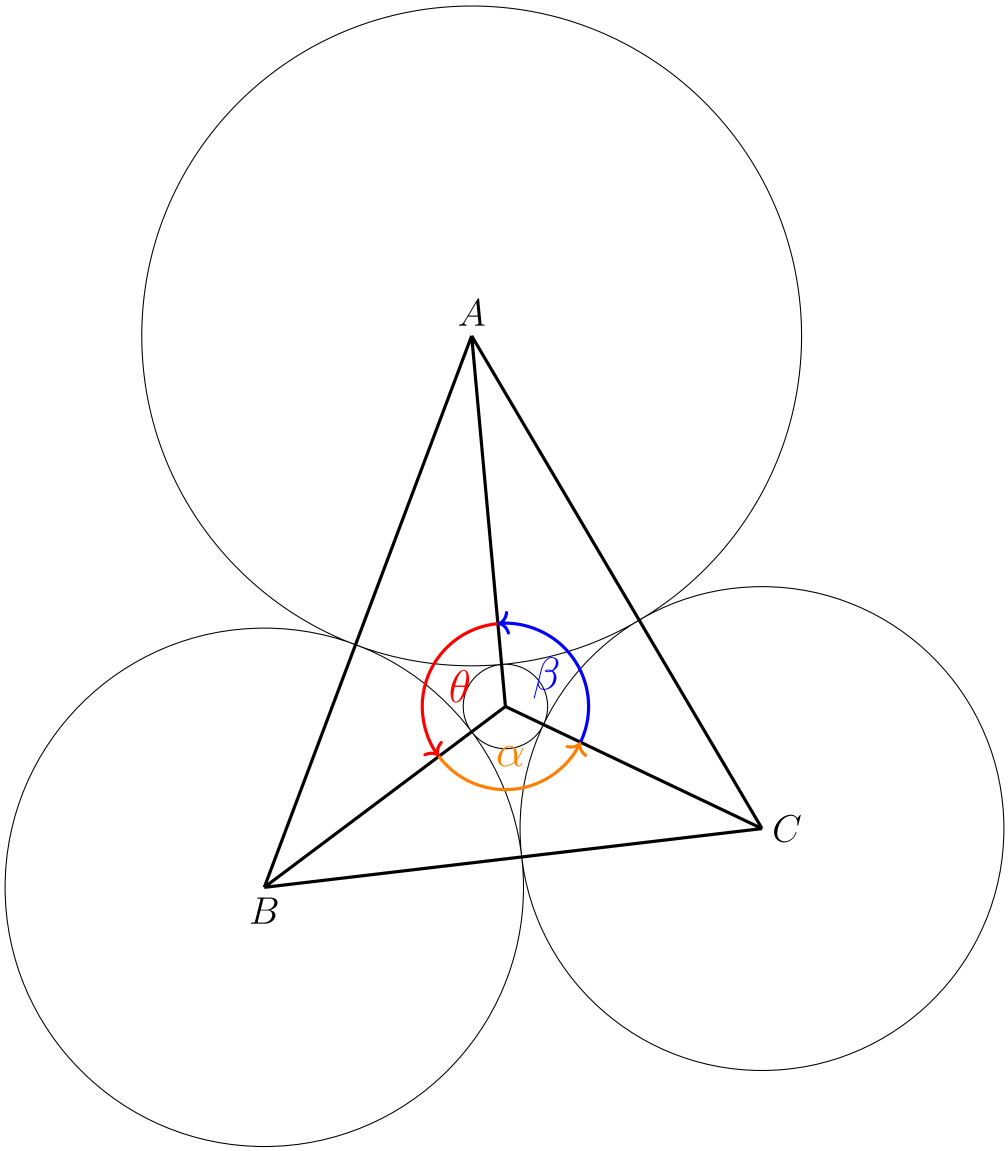


FIGURE 3. Four mutually tangent circles with centers A, B, C, and D.

Now we are ready to prove the theorem.

Proof. Suppose we have four mutually tangent circles with centers A, B, C, and D (refer to Figure 3 above) with respective radii r_A, r_B, r_C, and r_D. Circle D in the center is not labeled. The side lengths of $\triangle ABC$ are

$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad \text{and} \quad AC = r_A + r_C$$

and the lengths from the centers of circles A, B, C to D are

$$AD = r_A + r_D, \quad BD = r_B + r_D, \quad \text{and} \quad CD = r_C + r_D.$$

Let $\angle BDC = \alpha$, $\angle CDA = \beta$, and $\angle ADB = \theta$. The law of cosines in $\triangle ADB$ yields

$$\begin{aligned}
\cos\theta &= \frac{AD^2 + BD^2 - AB^2}{2 \cdot AD \cdot BD} \\
&= \frac{(r_A + r_D)^2 + (r_B + r_D)^2 - (r_A + r_B)^2}{2(r_A + r_D)(r_B + r_D)} \\
&= \frac{2r_D^2 + 2r_D(r_A + r_B) - 2r_A r_B}{2(r_A + r_D)(r_B + r_D)} \\
&= 1 - \frac{2r_A r_B}{(r_A + r_D)(r_B + r_D)}.
\end{aligned}$$

Similarly, we find in $\triangle ADB$ and $\triangle CDA$ that

$$\cos\alpha = 1 - \frac{2r_B r_C}{(r_B + r_D)(r_C + r_D)} \quad \text{and} \quad \cos\beta = 1 - \frac{2r_A r_C}{(r_A + r_D)(r_C + r_D)}.$$

Now replace each radius by its respective curvature k_A, k_B, k_C, and k_D and name the associated fraction to each angle λ

$$\begin{aligned}
\cos\alpha &= 1 - \frac{2k_D^2}{(k_B + k_D)(k_C + k_D)} = 1 - \lambda_\alpha \\
\cos\beta &= 1 - \frac{2k_D^2}{(k_A + k_D)(k_C + k_D)} = 1 - \lambda_\beta \\
\cos\theta &= 1 - \frac{2k_D^2}{(k_A + k_D)(k_B + k_D)} = 1 - \lambda_\theta.
\end{aligned}$$

By Lemma 3.2 we have that

$$\begin{aligned}
(1 - \lambda_\alpha)^2 + (1 - \lambda_\beta)^2 + (1 - \lambda_\theta)^2 &= 1 + 2(1 - \lambda_\alpha)(1 - \lambda_\beta)(1 - \lambda_\theta) \\
\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\theta^2 + 2\lambda_\alpha\lambda_\beta\lambda_\theta &= 2(\lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\theta + \lambda_\alpha\lambda_\theta) \\
\frac{\lambda_\alpha}{\lambda_\beta\lambda_\theta} + \frac{\lambda_\beta}{\lambda_\alpha\lambda_\theta} + \frac{\lambda_\theta}{\lambda_\alpha\lambda_\beta} + 2 &= 2\left(\frac{1}{\lambda_\alpha} + \frac{1}{\lambda_\beta} + \frac{1}{\lambda_\theta}\right).
\end{aligned}$$

Substituting back our values for the λs we find

$$\begin{gathered}
\frac{(k_A + k_D)^2}{2k_D^2} + \frac{(k_B + k_D)^2}{2k_D^2} + \frac{(k_C + k_D)^2}{2k_D^2} + 2 = \\
2\left(\frac{(k_B + k_D)(k_C + k_D)}{2k_D^2} + \frac{(k_A + k_D)(k_B + k_D)}{2k_D^2} + \frac{(k_A + k_D)(k_B + k_D)}{2k_D^2}\right).
\end{gathered}$$

We multiply through by $2k_d^2$ and simplfy to find that

$$\begin{aligned}
k_A^2 + k_B^2 + k_C^2 + 2k_D(k_A + k_B + k_C) + 7k_D^2 &= 6k_D^2 + 4k_D(k_A + k_B + k_C) \\
&\quad + 2(k_A k_B + k_B k_C + k_A k_C) \\
k_A^2 + k_B^2 + k_C^2 + k_D^2 &= 2k_D(k_A + k_B + k_C) \\
&\quad + 2(k_A k_B + k_B k_C + k_A k_C) \\
&= (k_A + k_B + k_C + k_D)^2 - (k_A^2 + k_B^2 + k_C^2 + k_D^2) \\
2(k_A^2 + k_B^2 + k_C^2 + k_D^2) &= (k_A + k_B + k_C + k_D)^2.
\end{aligned}$$

□

Remark. This proof is the exact same when one of the four tangent circles is the enclosing circle.

Now, beginning with three mutually tangent circles, we add the two circles of Apollonius. Repeating this process with the new triples that form, we create an *Apollonian circle packing*.

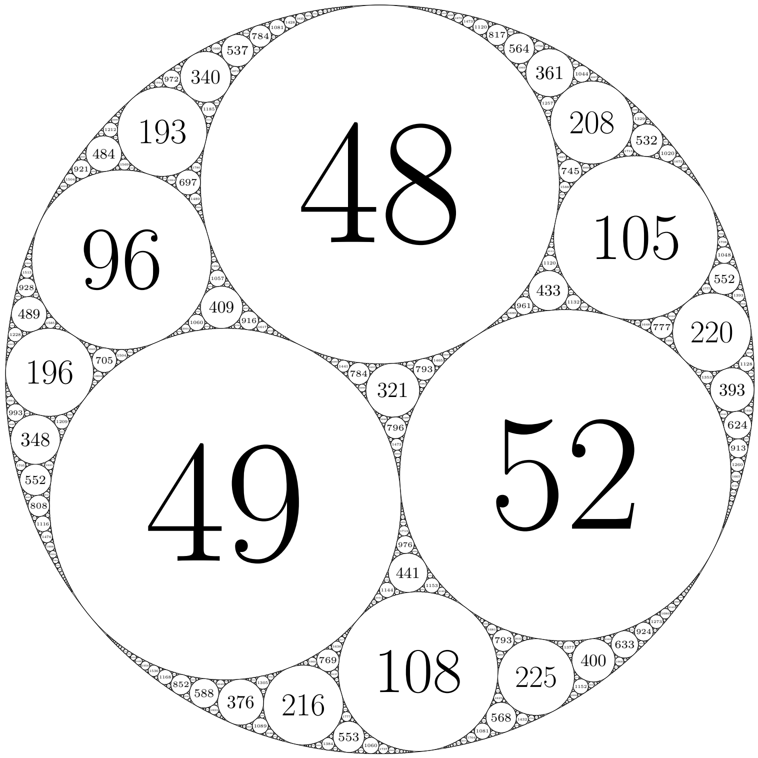


FIGURE 4. The Apollonian circle packing corresponding to $[-23, 48, 49, 52]$.

We can find the curvatures of the two circles of Apollonius in terms of the first three given circles' curvatures.

Corollary 3.3. *If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures*

$$d = a + b + c + 2\sqrt{ab + ac + bc} \quad \text{and} \quad d' = a + b + c - 2\sqrt{ab + ac + bc}.$$

Moreover, $d + d' = 2(a + b + c)$.

Proof. First, we solve for d from the Descartes Equation to find that

$$2\left(a^2 + b^2 + c^2 + d^2\right) - (a + b + c + d)^2 = 0$$
$$d^2 - 2d(a + b + c) + \left(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac\right) = 0.$$

The quadratic formula gives

$$\begin{aligned} d &= \frac{2(a + b + c) \pm \sqrt{4(a + b + c)^2 - 4\left(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac\right)}}{2} \\ &= a + b + c \pm 2\sqrt{ab + bc + ca}. \end{aligned}$$

Thus, there are two options for d. Their sum is $2(a + b + c)$. $\square$

This means that if a, b, c, $d \in \mathbb{Z}$ then the entire packing will consist of circles with integer curvatures. The question naturally arises: which integers will we see? We will choose primitive circle packings, that is, $\gcd(a, b, c, d) = 1$.

3.1. **The Apollonian Group.** Given a quadruple $[a, b, c, d]$ we obtain $[a, b, c, d']$ by matrix multiplication of

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

applied to the Descartes quadruple as a vector $[a, b, c, d]$. This comes from the equation for $d' = 2(a + b + c) - d$. Similarly, we define

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

Example 3.4. When we apply S_1 to the primitive quadruple $[-6, 11, 14, 15]$ we have

$$S_1 \begin{pmatrix} -6 \\ 11 \\ 14 \\ 15 \end{pmatrix} = \begin{pmatrix} -(-6) + 2(11 + 14 + 15) \\ 11 \\ 14 \\ 15 \end{pmatrix} = \begin{pmatrix} 86 \\ 11 \\ 14 \\ 15 \end{pmatrix}.$$

The example in Figure 1 shows this exact swap, just without curvatures labeled.

The *Apollonian Group* is given by the presentation $\mathcal{A} = \left\langle S_1, S_2, S_3, S_4 : S_i^2 = 1 \right\rangle$. Note that $\mathcal{A}$ is a subgroup of $O_{3,1}(\mathbb{Z})$ (the orthogonal group of a quadratic form of signature $3, 1$). In other words, $\mathcal{A}$ preserves the Descartes form. We can also write this as $S_i^T Q_D S_i = Q_D$ where

$$Q_D = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

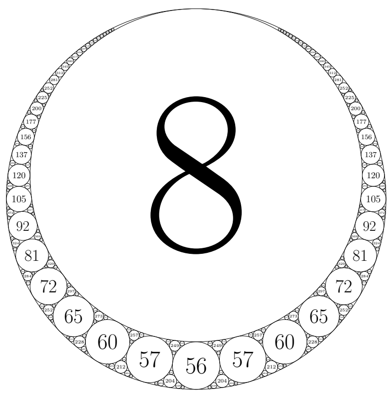
This also means that

$$\begin{pmatrix} a & b & c & d \end{pmatrix} Q_D \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \text{Descartes Form}.$$

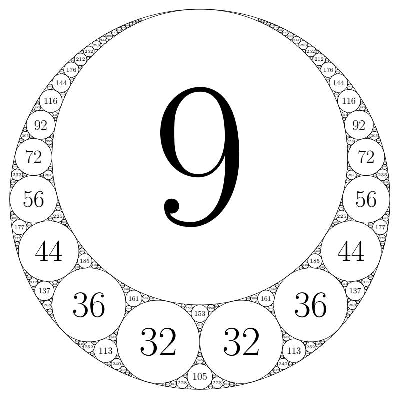
Like forms, Descartes quadruples also have a notion of reduction. We say a Descartes quadruple $[-a, b, c, d]$ is reduced if $-a \leqslant b \leqslant c \leqslant d$. By applying the above swaps, we can replace any curvature in a quadruple with a smaller curvature until we have the smallest possible four curvatures. As we saw above, $[86, 11, 14, 15]$ *reduces* to $[-6, 11, 14, 15]$.

4. Symmetric Packings

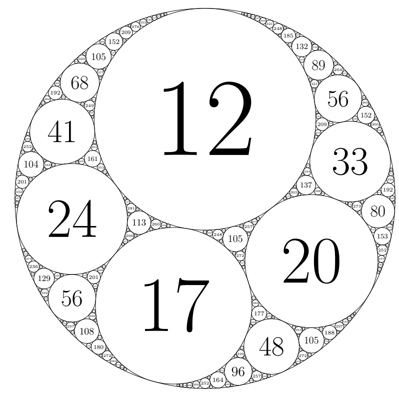
We say a positive integer a *has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$. So, it is natural to ask: given a positive integer n, what types of packings and how many does it have? For example, the integer 7 has precisely three packings:



$[-7, 8, 56, 57]$,



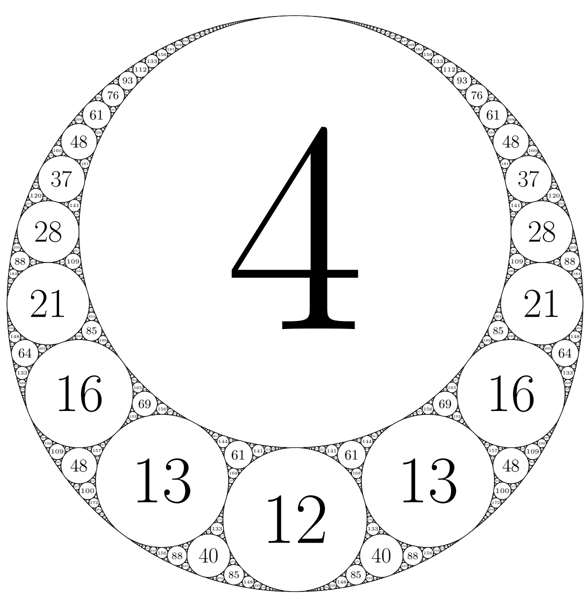
$[-7, 9, 32, 32]$,



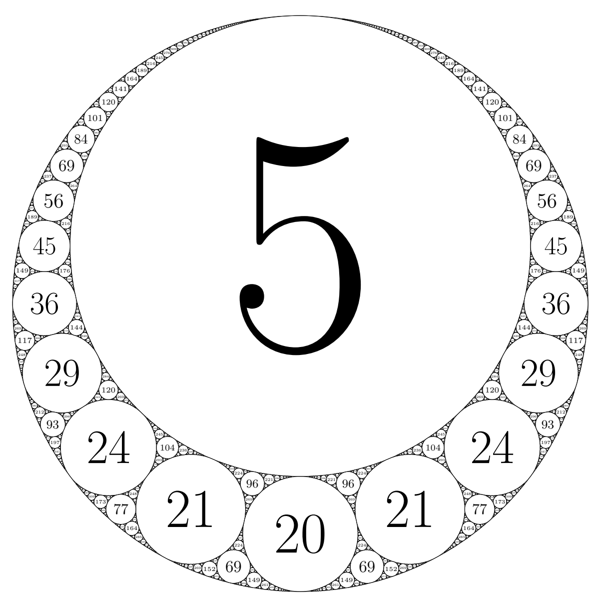
$[-7, 12, 17, 20]$.

Note the first two packings have a line of symmetry, whereas the third packing does not. In this section, we will focus on parameterizing and counting the number of each type of these packings that a given positive integer has. This will explain why 7 has no other packings. We will first look at the first two types of symmetric packings.

A packing is *symmetric* if it has a line of symmetry. A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$ (that is, the next circle generated after the first four has the same curvature as one of the first four). These packings have a line of symmetry that is not tangent to any circles. Two examples are:

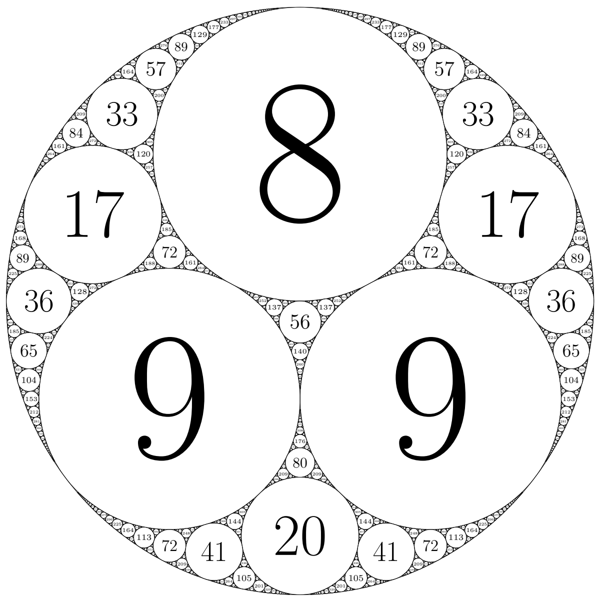


The sum-symmetric packing $[-3, 4, 12, 13]$.

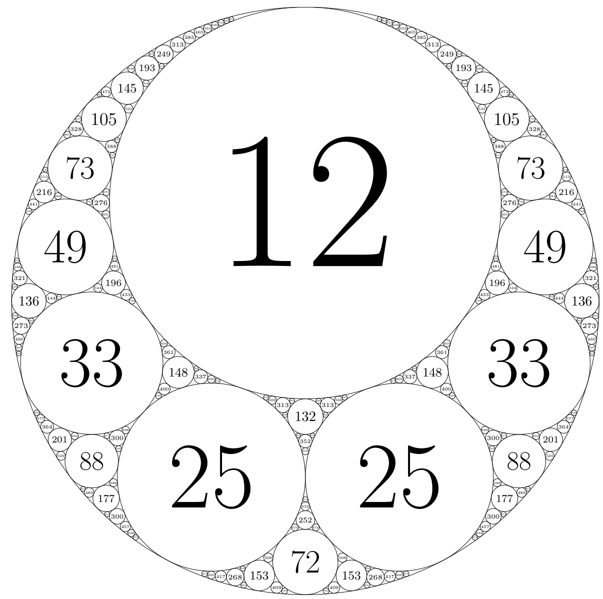


The sum-symmetric packing $[-4, 5, 20, 21]$.

A *twin-symmetric* quadruple is a primitive reduced Descartes quadruple with $c = d$ or $c = b$. These packings will have a line of symmetry tangent to the two circles with the same curvature. Two examples are:



The twin-symmetric packing $[-4, 8, 9, 9]$.



The twin-symmetric packing $[-8, 12, 25, 25]$.

There are two other packings that fit into both of these categories, the first is the strip packing which is an infinite packing with infinitely many lines of symmetry! We will prove this later in Proposition 4.3.

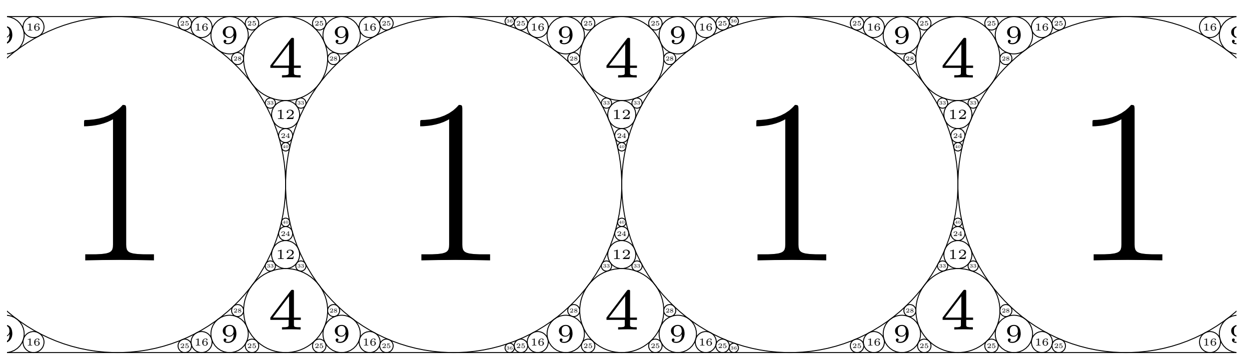


Figure 8. The strip packing: $[0, 0, 1, 1]$.

The second is the so called *bug-eye* packing, which has a line of symmetry corresponding to a sum-symmetric packing and one corresponding to a twin-symmetric packing.

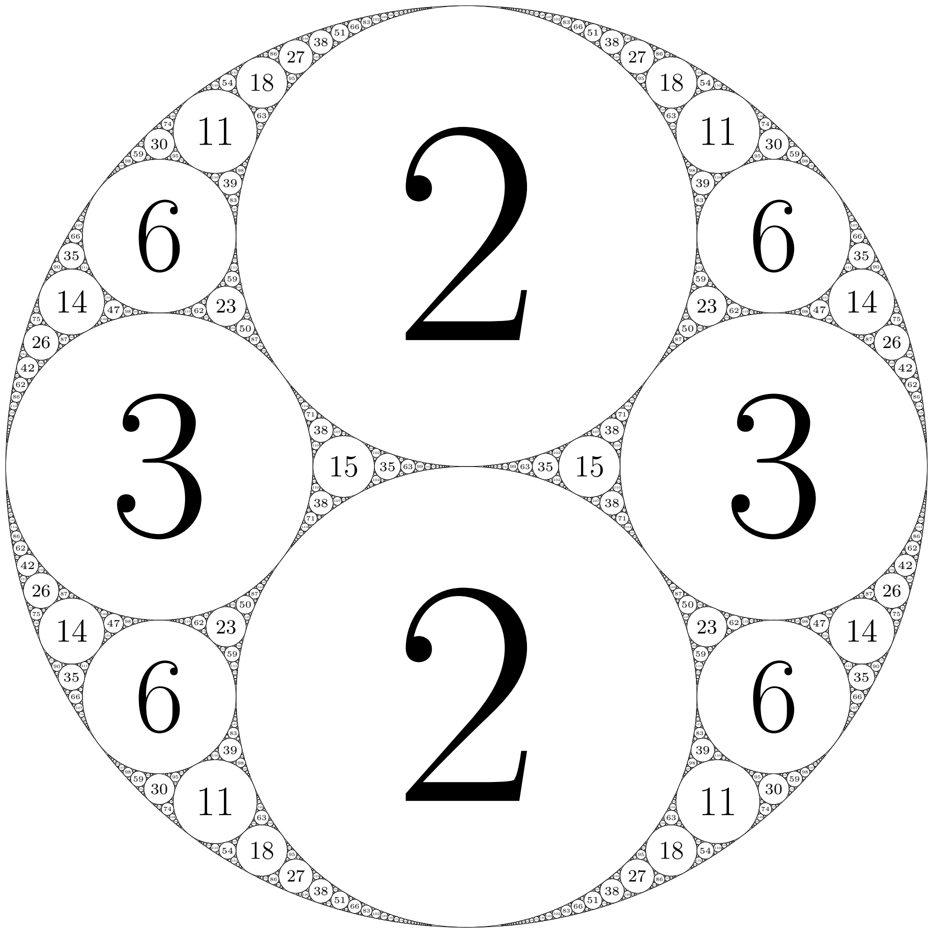


Figure 9. The bug-eye packing: $[-1, 2, 2, 3]$.

We will now prove some important properties of sum-symmetric packings.

Proposition 4.1. *The following equalities hold in a sum-symmetric packing* $[a, b, c, d]$.

(i) $a + b = d - c$,
(ii) $d^2 = a^2 + b^2 + c^2$, *and*
(iii) $ab + ac + bc = 0$.

Proof.

(i) We know that a sum-symmetric packing has the property that $2(a + b + c) - d = d$ which immediately yields $a + b = d - c$.
(ii) Plugging part (i) back into the Descartes Equation we find

$$\begin{aligned}(a+b+c+d)^2 &= 2\left(a^2+b^2+c^2+d^2\right)\\(d-c+c+d)^2 &= 2a^2+2b^2+2c^2+2d^2\\4d^2 &= 2\left(a^2+b^2+c^2\right)+2d^2\\d^2 &= a^2+b^2+c^2.\end{aligned}$$

(iii) Use substitutions from parts (i) and (ii) to find

$$\begin{aligned} a+b+c &= d \\ (a+b+c)^2 &= a^2+b^2+c^2 \\ a^2+b^2+c^2+2ab+2ac+2bc &= a^2+b^2+c^2 \\ ab+ac+bc &= 0. \end{aligned}$$

□

We will now prove these are the only two types of symmetric packings.

Proposition 4.2. *A symmetric packing (excluding the strip packing) is either sum-symmetric or twin-symmetric.*

Proof. There are number of requirements for the packing of a given reduced Descartes quadruple (A, B, C, D) to have a line of symmetry. The first is that the line must go through the center of the enclosing circle, that is, the line of symmetry must be a diameter of A. The line must also be a line of symmetry of B, C, or D. Without loss of generality, take the line of symmetry through B.

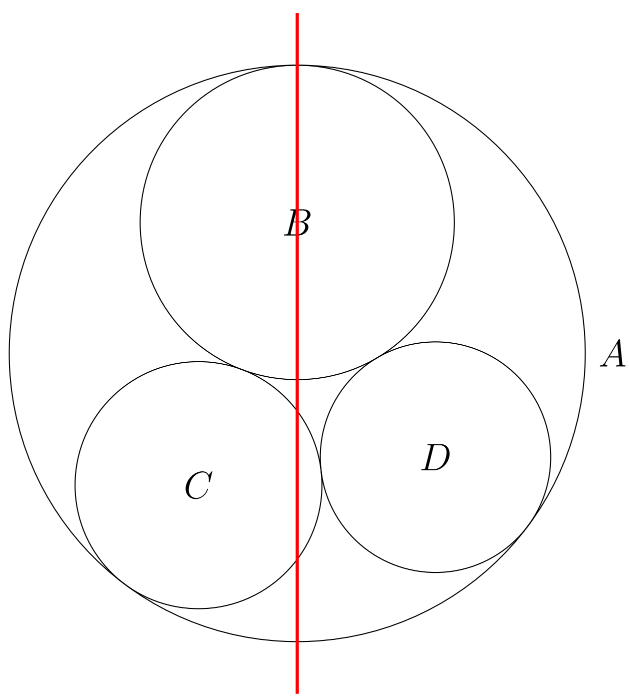


FIGURE 10. Possible line of symmetry for the quadruple (A, B, C, D).

Figure 10 demonstrates that once a possible line has been selected there are two possibilities for the packing to be symmetric: either C and D must be reflections of one another or the line is also a line of symmetry of one of them. In the second case, D' and D must be reflections. Note that in a reduced quadruple, two circles cannot be on the same side of this line, else they leave room for a larger circle and thus a smaller curvature.

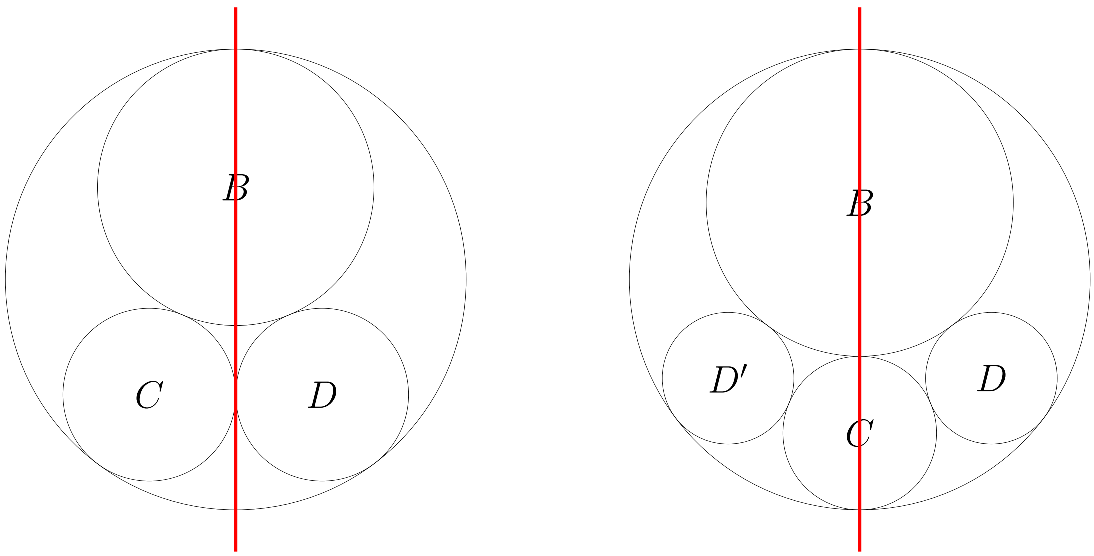


FIGURE 11. The two possibilities for the packing of a quadruple to be symmetric.

Thus, if a packing is symmetric, either C is a reflection of D and thus their curvatures are the same so the packing is twin-symmetric or D is a reflection of D' so the packing is sum-symmetric. □

Proposition 4.3. *Only the strip and bug-eye packing are both sum-symmetric and twin-symmetric.*

Proof. Suppose we have a sum-symmetric packing $[a, b, c, d]$, that is $a + b + c = d$. If $c = d$ then

$$\begin{aligned} a + b + c &= c \\ a + b &= 0. \end{aligned}$$

Plugging $a + b = 0$ and $c = d$ into the Descartes Equation we have

$$\begin{aligned} 2(a^2 + b^2 + c^2 + d^2) &= (a + b + c + d)^2 \\ 2(a^2 + b^2 + 2c^2) &= (c + c)^2 \\ 2a^2 + 2b^2 + 4c^2 &= 4c^2 \\ 2a^2 + 2b^2 &= 0 \end{aligned}$$

which only has the integer solution $a = b = 0$, which gives the strip packing $[0, 0, 1, 1]$. If $b = c$ then

$$\begin{aligned} a + c + c &= d \\ a + 2c &= d \\ a^2 + 4ac + 4c^2 &= d^2, \end{aligned}$$

and by part (ii) of p roposition 4.1 we have

$$\begin{aligned} d^2 &= a^2 + b^2 + c^2 \\ a^2 + 4ac + 4c^2 &= a^2 + c^2 + c^2 \\ 4ac + 2c^2 &= 0 \\ c(2a + c) &= 0 \end{aligned}$$

which means either $c = 0$ which is impossible, or $2a + c = 0$ which means $c = -2a$. Now we find

$$d = a + b + c = a + -2a - 2a = -3a$$

which gives the packing $[a, -2a, -2a, -3a]$. This is only primitive and reduced when $a = -1$ which gives the bug eye packing $[-1, 2, 2, 3]$. □

Theorem 4.4. *A sum-symmetric quadruple* $[a, b, c, d]$ *is of the form*

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

with $\gcd(x, y) = 1$*, and* $x, y \geqslant 0$*.*

Proof. Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition 4.1 (iii) we have

$$\begin{aligned} ab + ac + bc &= 0 \\ a^2 + ab + ac + bc &= a^2 \\ (a+b)(a+c) &= a^2. \end{aligned}$$

Let $g = \gcd(a+b, a+c)$ so that $a+b = gx^2$ and $a+c = gy^2$ for some $x, y \geqslant 0$. This yields $gxy = -a$ as $+a$ gives a non reduced quadruple. Now, we have

$$b = (a+b) + (-a) = gx^2 + gxy \quad \text{and} \quad c = (a+c) + (-a) = gy^2 + gxy.$$

Using the relation $d = a + b + c$ we can substitute what we have just found to find

$$d = a + b + c = (-gxy) + (gx^2 + gxy) + (gy^2 + gxy) = g((x+y)^2 - xy).$$

Thus, we have found that a sum-symmetric quadruple is given by

$$\left[-gxy,\ gx(x+y),\ gy(x+y),\ g((x+y)^2 - xy).\right]$$

Clearly, for the quadruple to be primitive, g must be 1, meaning x and y are coprime. Thus,

$$a = -xy,\ b = x(x+y),\ c = y(x+y),\ \text{and } d = (x+y)^2 - xy$$

with $\gcd(x, y) = 1$. □

This means that the number of sum-symmetric packings of n is the same as the number of coprime factor pairs of n. We define

$$\omega(n) = \begin{cases} \#\{\text{distinct prime divisors of } n\} & n \geqslant 2 \\ 1 & n = 1. \end{cases}$$

Corollary 4.5. *A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings.*

Proof. Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy but we divide by two to account for symmetry. Thus, n has $2^k/2 = 2^{k-1} = 2^{\omega(n)-1}$ sum-symmetric packings. □

Proposition 4.6. *In a reduced Descartes quadruple* $[a, b, c, d]$*, two of* a, b, c, d *are odd and two of* a, b, c, d *are even.*

Proof. Clearly all of a, b, c, d cannot be even else the quadruple is not reduced. Taking the Descartes equation mod 2 gives $a^2 + b^2 + c^2 + d^2 \equiv 0 \bmod 2$. As squaring preserves parity, this means $a + b + c +$

$d = 0 \bmod 2$, so it is impossible that exactly one of a, b, c, or d is even or odd. To rule out the case that all of them are odd, take the Descartes equation mod 16 to find that $(a+b+c+d)^2 \equiv 8 \bmod 16$ which is impossible in integers. Therefore, two of a, b, c, d are odd and two of a, b, c, d are even. $\square$

Corollary 4.7. *In a twin-symmetric packing $[a, b, c, c]$, a and b have the same parity.*

Now we are ready to parameterize all twin-symmetric packings.

Theorem 4.8. *A twin-symmetric quadruple is one of following two forms*

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Proof. Without loss of generality, we will consider the cases $c = d$ and $c = b$ to be the same as they are both reduced. That is, if $c = b$, switching b and d does not change if the quadruple is reduced or not. Now, plugging in $c = d$ into the Descartes Equation gives

$$\begin{aligned} (a+b+c+c)^2 &= 2(a^2+b^2+c^2+c^2) \\ 2ab+4ac+4bc &= a^2+b^2 \\ 4(ac+bc) &= a^2-2ab+b^2 \\ c(a+b) &= \left(\frac{b-a}{2}\right)^2. \end{aligned}$$

Because a and b have the same parity by Corollary 4.7 and $a < b$, $\frac{b-a}{2} \in \mathbb{N}$. So, let $g = \gcd(a+b, c)$ so that $c = gm^2$ and $a + b = gn^2$ for some $m, n \geqslant 0$. This yields $gmn = \frac{b-a}{2}$, as $\frac{a-b}{2}$ gives a non reduced quadruple. Hence,

$$m = \sqrt{\frac{c}{g}} \quad \text{and} \quad n = \sqrt{\frac{a+b}{g}}.$$

Case 1. When a and b are both odd, c must be even by Corollary 4.7. Thus, $g = \gcd(a+b, c)$ is even, so $g = 2k$ for some $k \in \mathbb{N}$. This yields

$$m = \sqrt{c/2k} \quad \text{and} \quad n = \sqrt{(a+b)/2k}.$$

Now, let $x = 2m - n$ and $y = n$ so that

$$-xy = -(2m-n)n = -2mn + n^2 = -\frac{b-a}{2k} + \frac{a+b}{2k} = \frac{a}{k}.$$

This means $a = -kxy$. Now, we obtain expressions for b and c as follows:

$$b = -a + (a+b) = -(-kxy) + 2kn^2 = kxy + 2ky^2 = k(xy+2y^2)$$

and

$$\begin{aligned} c = 2k\frac{c}{2k} = k\frac{4m^2}{2} &= k\frac{4m^2-4mn+n^2+4mn-2n^2+n^2}{2} \\ &= k\frac{(2m-n)^2+2n(2m-n)+n^2}{2} = k\frac{x^2+2xy+y^2}{2} = k\frac{(x+y)^2}{2}. \end{aligned}$$

Thus, we have found that when a and b are odd, the twin-symmetric packings is given by

$$\left[-kxy,\ k(xy+2y^2),\ \tfrac{k}{2}(x+y)^2,\ \tfrac{k}{2}(x+y)^2\right].$$

Clearly, for the quadruple to be primitive, k must be 1, so $g = 2$ and x and y are coprime, which gives the parameterization

$$\left[-xy,\ xy + 2y^2,\ \tfrac{1}{2}(x+y)^2,\ \tfrac{1}{2}(x+y)^2\right].$$

Note that $x + y = 2m - n + n = 2m$ which is even so $\frac{1}{2}(x + y)$ is an integer. With these choices, it easy to check this quadruple is always primitive.

Case 2. When a and b are both even, c must be odd by Corollary 4.7. Thus, $g = \gcd(a + b, c)$ is odd. This yields

$$m = \sqrt{c/g} \quad \text{and} \quad n = \sqrt{(a+b)/g},$$

where n is even. Now, let $x = m - n/2$, and $y = n/2$ which are both positive, so that

$$-2xy = -2\left(m - \frac{n}{2}\right)\left(\frac{n}{2}\right) = -mn + \frac{n^2}{2} = \frac{b-a}{2g} + \frac{a+b}{2g} = \frac{a}{g}$$

which means $a = -2gxy$. Now, we obtain expressions for b and c as follows:

$$b = -a + (a + b) = -(-2gxy) + gn^2 = 2gxy + 4gy^2 = g\left(2xy + 4y^2\right)$$

and

$$\begin{aligned} c = gm^2 &= g\left(m^2 - mn + \frac{n^2}{4} + mn - \frac{n^2}{2} + \frac{n^2}{4}\right) \\ &= g\left(\left(m - \frac{n}{2}\right)^2 + 2\left(m - \frac{n}{2}\right)\left(\frac{n}{2}\right) + \frac{n^2}{4}\right) = g(x^2 + xy + y^2) = g(x+y)^2. \end{aligned}$$

Thus, we have found that when a and b are even, the twin-symmetric packings is given by

$$\left[-2gxy,\ g(2xy + 4y^2),\ g(x+y)^2,\ g(x+y)^2\right].$$

For the quadruple to be primitive, $g = 1$, so x and y are coprime, which gives the parameterization

$$\left[-2xy,\ 2xy + 4y^2,\ (x+y)^2,\ (x+y)^2\right].$$

With these choices, it easy to check this quadruple is always primitive. ☐

To state the number of twin-symmetric packing that n has, we define δ_n as in [GLM⁺03]:

$$\delta_n = \begin{cases} 1 & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 4.9. *A natural number n has $(1 - \delta_n) \cdot 2^{\omega(n)-1}$ twin-symmetric packings where $\omega(n)$ is the number of distinct prime divisors of n.*

Proof. When $n \equiv 2 \bmod 4$, write $n = 2 \cdot p_1 \cdot p_2 \cdots p_k$, where p_i are odd primes. There are no factor pairs of n with both factors odd, so the first parameterization is impossible. When x is even $4 \mid 2xy$ and hence $2xy \not\equiv 2 \bmod 4$, so the second parameterization is also impossible. Therefore, any natural number equivalent to 2 mod 4 has no symmetric packings.

The proof for when $n \not\equiv 2 \bmod 4$ follows exactly the same as Corollary 4.5. ☐

Corollary 4.10. *Every $n \in \mathbb{N} \geqslant 3$ has at least one sum-symmetric and one twin-symmetric packing given by*

$$[-n,\ n+1,\ n^2+n,\ n^2+n+1]$$

and

$$\begin{cases} \left[-n,\ n+2,\ 2\left(\dfrac{n+1}{2}\right)^2,\ 2\left(\dfrac{n+1}{2}\right)^2\right] & n \text{ odd} \\ \left[-n,\ n+4,\ \left(\dfrac{n+2}{2}\right)^2,\ \left(\dfrac{n+2}{2}\right)^2\right] & n \equiv 0 \pmod 4 \\ \left[-n,\ n+4,\ \dfrac{n}{2}\left(\dfrac{n+4}{2}\right),\ \dfrac{n}{2}\left(\dfrac{n+4}{2}\right)+4\right] & n \equiv 2 \pmod 4, \end{cases}$$

respectively.

Theorem 4.11 ([GLM+03, Theorem 4.2])**.** *The primitive, reduced Descartes quadruples $[-a, b, c, d]$ with $-a < 0 \leqslant b \leqslant c \leqslant d$ are in one-to-one correspondence with positive definite forms of discriminant $-4a^2$ having nonnegative middle coefficient. The associated reduced binary quadratic form $[A, B, C] = Ax^2 + Bxy + Cy^2$ is given by*

$$[A, B, C] := [-a+b, -a+b+c-d, -a+c]\,.$$

Primitive root quadruples correspond to reduced binary quadratic forms having a nonnegative middle coefficient. In particular, the number of primitive root quadruples of $-a$ is equal to $h\left(-4a^2\right)$, the number of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of positive definite forms of discriminant $-4a^2$.

This theorem allows us to directly convert a Descartes quadruple to a reduced form.

Theorem 4.12. *The corresponding quadratic form of a sum-symmetric quadruple*

$$[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy]$$

is given by

$$Q = \left[x^2, 0, y^2\right].$$

Proof. Plugging in the corresponding values into Theorem 4.11 we find

$$\begin{aligned} Q(u,v) &= [A, B, C] = [-a+b, -a+b+c-d, -a+c] \\ &= \left[-xy + (xy + x^2), -xy + (xy + x^2) + (xy + y^2) - \left((x+y)^2 - xy\right), -xy + (xy + y^2)\right] \\ &= \left[x^2, 0, y^2\right]. \end{aligned}$$

∎

Theorem 4.13. *The corresponding quadratic form of a twin-symmetric quadruple*

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & x \text{ even} \quad x > y \end{cases}$$

is given by

$$Q = \left[2y^2, 2y^2, \tfrac{1}{2}(x^2+y^2)\right] \quad \text{and} \quad Q = \left[4y^2, 4y^2, x^2+y^2\right],$$

respectively.

Proof. Plugging the first paramaterization's values into Theorem 4.11 we find

$$\begin{aligned}
Q &= [A,B,C] = [-a+b, -a+b+c-d, -a+c] \\
&= \left[-xy+(xy+2y^2), -xy+(xy+2y^2)+\tfrac{1}{2}(x+y)^2-\tfrac{1}{2}(x+y)^2, -xy+\left(\tfrac{1}{2}(x+y)^2\right)\right] \\
&= \left[2y^2, 2y^2, \tfrac{1}{2}(x^2+y^2)\right].
\end{aligned}$$

Plugging the second paramaterization's values into Theorem 4.11 we find

$$\begin{aligned}
Q &= [A,B,C] = [-a+b, -a+b+c-d, -a+c] \\
&= \left[-2xy+2xy+4y^2, -2xy+2xy+4y^2+(x+y)^2-(x+y)^2, x^2+y^2\right] \\
&= \left[4y^2, 4y^2, x^2+y^2\right].
\end{aligned}$$

□

These forms are indeed the ambiguous forms of $\mathrm{cl}(-4a^2)$ which correspond to the reduced forms $|B| \leqslant A \leqslant C$. They arise on the "edge cases:"

(1) $B = 0$,
(2) $|B| = A$, or
(3) $A = C$.

and correspond to the forms with order in the class group at most two. See Corollary 4.9 and pages 7-8 in [Bue89].

These results also align with those in elementary genus theory. The following is Proposition 3.11 in [Cox13]:

Proposition 4.14. *Let $D \equiv 0, 1 \bmod 4$ be negative, and let r be the number of distinct odd primes dividing D. Define the number μ as follows: if $D \equiv 1 \bmod 4$, then $\mu = r$, and if $D \equiv 0 \bmod 4$, then $D = -4n$, where $n > 0$, and μ is determined by the following table:*

n	μ
$n \equiv 3 \bmod 4$	r
$n \equiv 1, 2 \bmod 4$	$r+1$
$n \equiv 4 \bmod 8$	$r+1$
$n \equiv 0 \bmod 8$	$r+2$

Then the class group $cl(D)$ has exactly $2^{\mu-1}$ elements of order $\leqslant 2$.

5. Non-Symmetric Packings

As opposed to symmetric-packings, non-symmetric packings are far more common. They do not have simple parameterizations like the sum-symmetric or twin-symmetric packings. Instead they are best understood as infinite families as found by [BTK11]. The following graph shows the total number of non-symmetric packings of each number up to $100,000$.

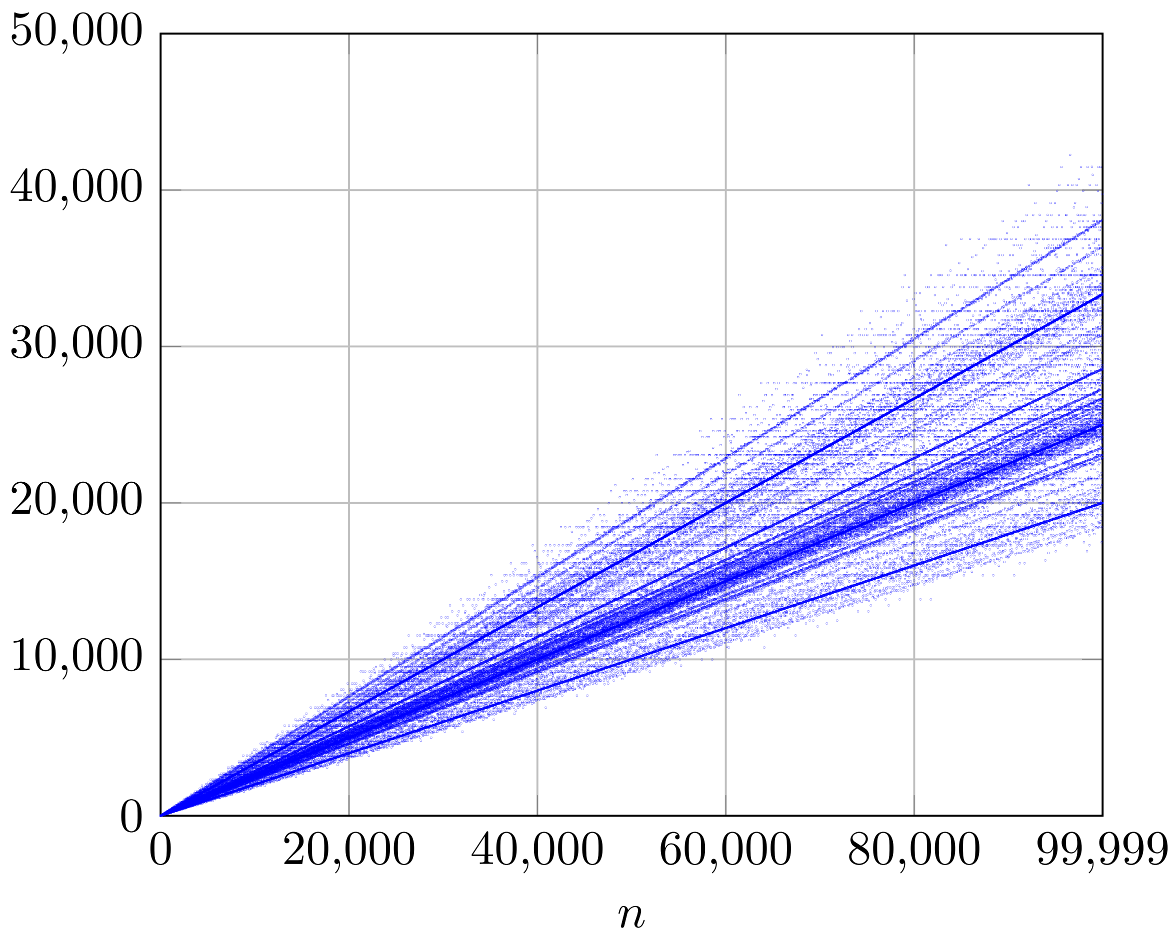


FIGURE 12. The number of non-symmetric packings of n up to $100,000$.

However, there are some families of non-symmetric packings based on modular criteria. For example, if $n \equiv 0 \bmod 3$, then

$$\left[-n,\, n+9,\, \frac{n^2}{9}+n+1,\, \frac{n^2}{9}+n+4\right]$$

is a Descartes quadruple. If $n \equiv 1 \bmod 5$, then

$$\left[-n,\, n+5,\, \frac{n^2+4}{5}+1,\, \frac{n^2+4}{5}+2\right]$$

is a Descartes Quadruple.

In [BTK11], the authors provide parameterizations of non-symmetric packings and show that every primitive reduced Descartes Quadruple (symmetric or not) can be written as

$$\left[-n,\, n+k,\, \frac{n^2+kn+\alpha^2}{k},\, \frac{n^2+kn+(k-\alpha)^2}{k}\right]$$

where n, k, and α are integers such that $\frac{n^2+\alpha^2}{k} \in \mathbb{N}, \gcd\left(n, k, \frac{n^2+\alpha^2}{k}\right) = 1, n, k > 0$, and $0 \leqslant \alpha \leqslant (k-\alpha)$. They also proved that given two packings of n,

$$\left[-n,\, n+k,\, \frac{n^2+kn+\alpha_k^2}{k},\, \frac{n^2+kn+(k-\alpha_k)^2}{k}\right] \quad \text{where } n \equiv n_k \pmod{k}$$

and

$$\left[-n,\, n+k',\, \frac{n^2+k'n+\alpha_{k'}^2}{k'},\, \frac{n^2+k'n+(k'-\alpha_{k'})^2}{k'}\right] \quad \text{where } n \equiv n_{k'} \pmod{k'},$$

where k and k' are relatively prime, we can generate an associated composition:

$$\left[-n,\, n+kk',\, \frac{n^2+kk'n+\alpha_{kk'}^2}{kk'},\, \frac{n^2+kk'n+(kk'-\alpha_{kk'})^2}{kk'}\right].$$

Theorem 5.1 ([GLM+03, Theorem 4.3])**.** *A natural number $n \geqslant 2$ has*

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(n)-\delta_n-1}$$

primitive reduced quadruples, where $\chi_{-4}(n) = (-1)^{(n-1)/2}$ for odd n and 0 for even n.

Corollary 5.2. *The number of non-symmetric packings of n is given by*

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right) + \left(2^{\omega(n)-1}\right)\left(2^{-\delta_n}-2+\delta_n\right).$$

Proof. We simply subtract the sum-symmetric and twin-symmetric packings from the total number of packings given by Theorem 5.1. This gives

$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(n)-\delta_n-1} - \underbrace{(1-\delta_n)\cdot 2^{\omega(n)-1}}_{\text{twin-symmetric}} - \underbrace{2^{\omega(n)-1}}_{\text{sum-symmetric}}$$
$$= \frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right) + \left(2^{\omega(n)-1}\right)\left(2^{-\delta_n}-2+\delta_n\right). \qquad \square$$

6. Extended Example

Let's use our results to find every packing of the number $20 = 2^2 \cdot 5$. As 20 has two distinct prime divisors, $\omega(20) = 2$ and $20 \not\equiv 2 \bmod 4$, so $\delta_{20} = 0$. Theorem 5.1 tells us that the total number of packings of 20 is

$$\frac{20}{4}\prod_{p|20}\left(1-\frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20)-\delta_{20}-1} = 5\left(1-\frac{\chi_{-4}(2)}{2}\right)\left(1-\frac{\chi_{-4}(5)}{5}\right) + 2^{2-0-1}$$
$$= 5\left(1-\frac{0}{2}\right)\left(1-\frac{1}{5}\right) + 2 = 5\left(\frac{4}{5}\right) + 2 = 6.$$

Corollary 4.5 says that of these 6 total packings, $2^{\omega(20)-1} = 2$ are sum-symmetric packings. Corollary 4.9 says that $(1-\delta_{20})2^{\omega(20)-1} = 2$ are twin-symmetric. Thus, we know 20 has two sum-symmetric packings, two twin-symmetric packings, and two non-symmetric packings. Let's find exactly what these packings are. We know the two coprime factor pairs of 20 are $(1, 20)$ and $(4, 5)$. We plug these into Theorem 4.4 which yields the two packings

$$(1,20) \implies \left[-1\cdot 20,\ 1(1+20),\ 20(1+20),\ (1+20)^2 - 1\cdot 20\right] = [-20,\ 21,\ 420,\ 421]$$
$$(4,5) \implies \left[-4\cdot 5,\ 4(4+5),\ 5(4+5),\ (4+5)^2 - 4\cdot 5\right] = [-20,\ 36,\ 45,\ 61]\,.$$

Now, plugging them into Theorem 4.8 (and dividing the even term by two yields the two packings

$$(1,10) \implies \left[-2\cdot 1\cdot 10,\ 2\cdot 1\cdot 10 + 4(1)^2,\ (1+10)^2,\ (1+10)^2\right] = [-20,\ 24,\ 121,\ 121]$$
$$(2,5) \implies \left[-2\cdot 2\cdot 5,\ 2\cdot 2\cdot 5 + 4(2)^2,\ (2+5)^2,\ (2+5)^2\right] = [-20,\ 36,\ 49,\ 49]\,.$$

To find the non-symmetric packings, we note $20 \equiv 7 \bmod 13$ which corresponds to the family

$$\left[-n,\ n+13,\ \left(\frac{n^2+13n+4^2}{13}\right),\ \left(\frac{n^2+13n+(13-4)^2}{13}\right)\right]$$

so

$$\left[-20,\ 20+13,\ \left(\frac{20^2+13\cdot 20+4^2}{13}\right),\ \left(\frac{20^2+13\cdot 20+(13-4)^2}{13}\right)\right] = [-20,\ 33,\ 52,\ 57]$$

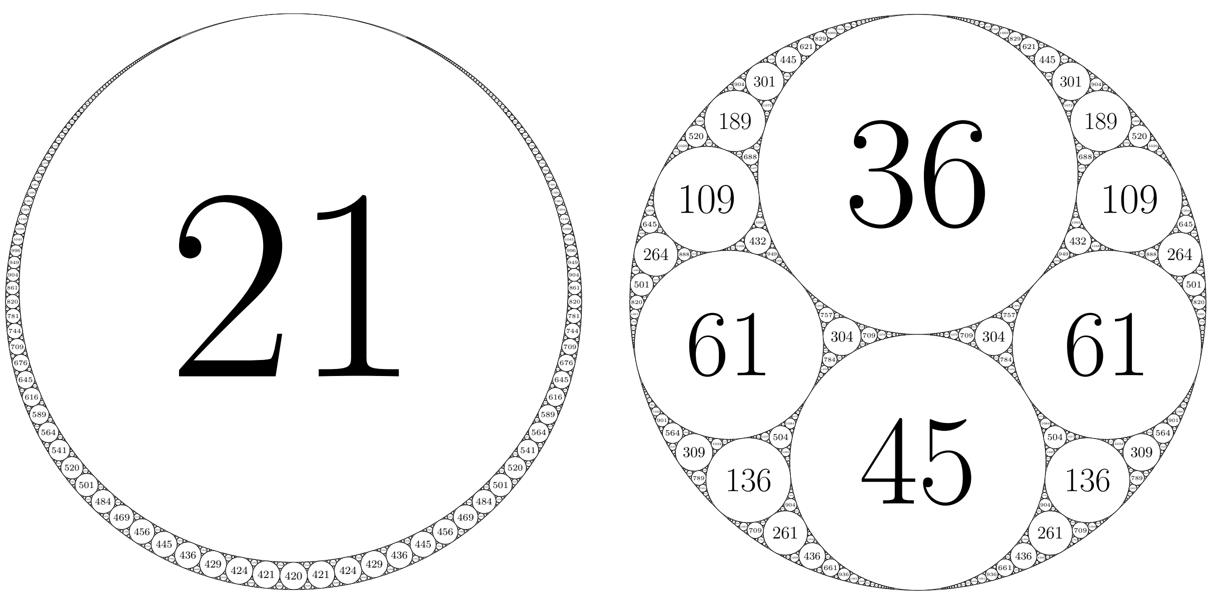
is a non-symmetric packing. Additionally, $20 \equiv 3 \bmod 17$ which corresponds to the family

$$\left[-n,\ n+17,\ \left(\frac{n^2+17n+5^2}{17}\right),\ \left(\frac{n^2+17n+(17-5)^2}{17}\right)\right]$$

so

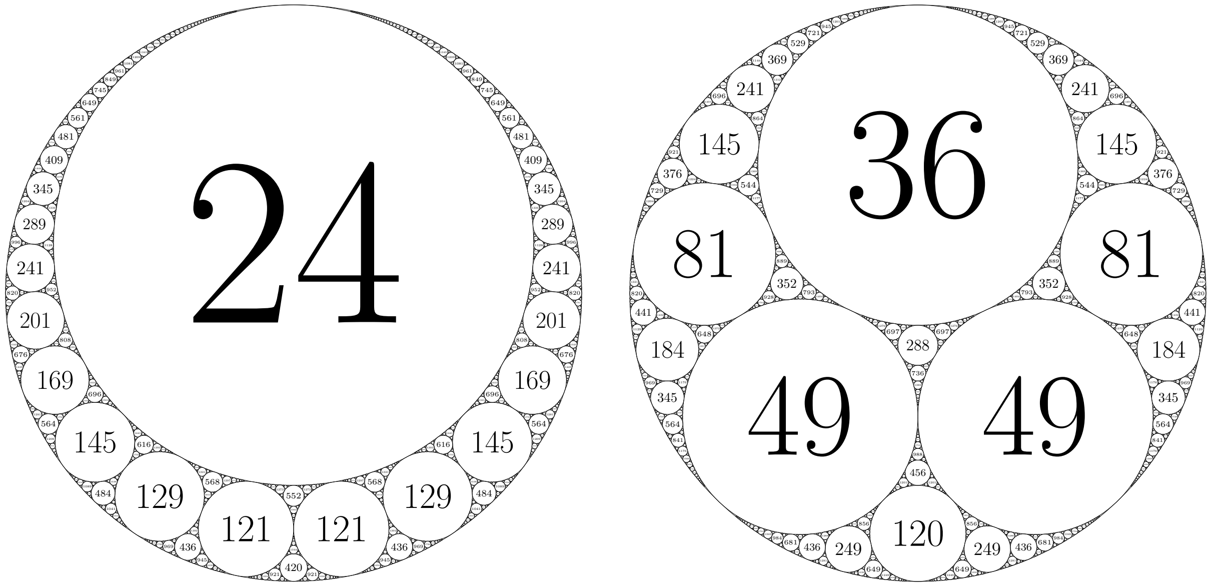
$$\left[-20,\ 20+17,\ \left(\frac{20^2+17\cdot 20+5^2}{17}\right),\ \left(\frac{20^2+17\cdot 20+12^2}{17}\right)\right] = [-20,\ 37,\ 45,\ 52]\,.$$

Thus, we have found and categorized all 6 packings that 20 has. We summarize our findings with the display:



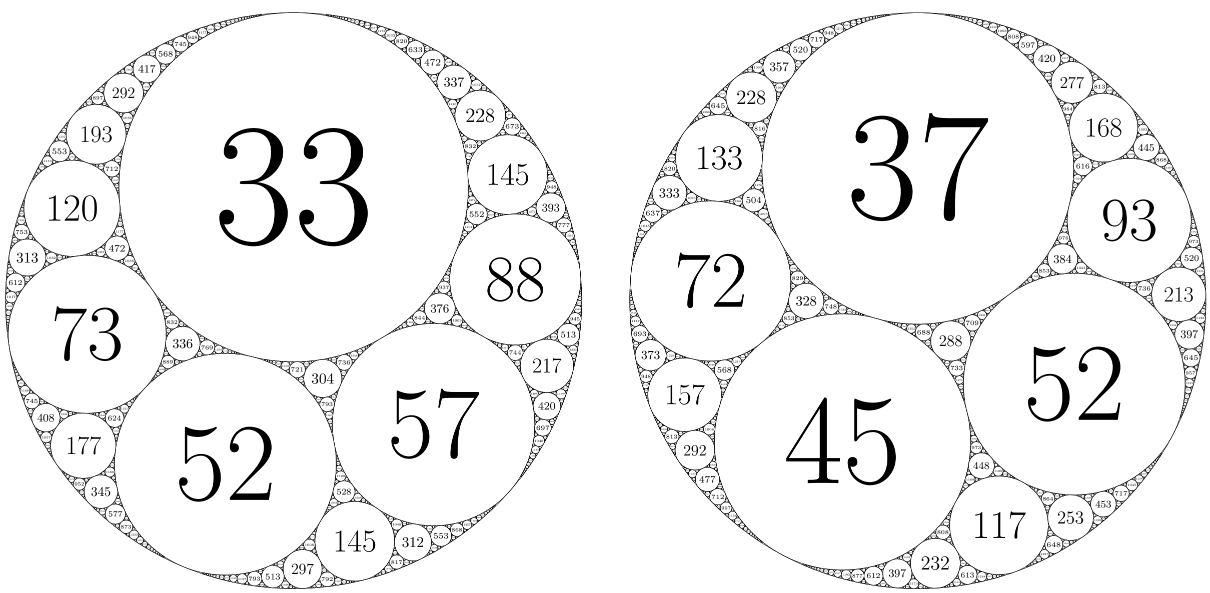
[−20, 21, 420, 421].   [−20, 36, 45, 61]

Figure 13. The two sum-symmetric packings of 20.



[−20, 24, 121, 121] [−20, 36, 49, 49].

Figure 14. The two twin-symmetric packings of 20.



[−20, 33, 52, 57]. [−20, 37, 45, 52]

Figure 15. The two non-symmetric packings of 20.

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