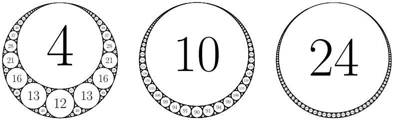
Apollonian Circle Packings & Parameterizations of Descartes Quadruples

Clyde Kertzer

University of Colorado Boulder



(Wednesday, April 10, 2024)

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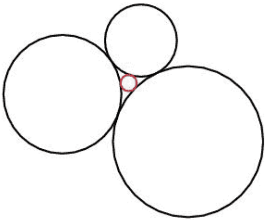

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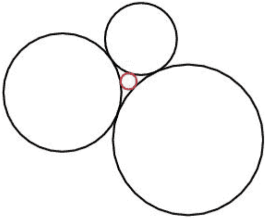

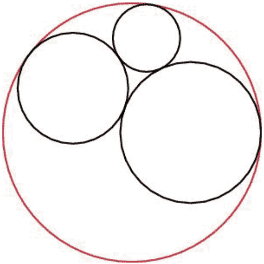

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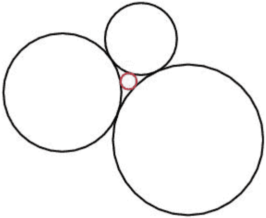

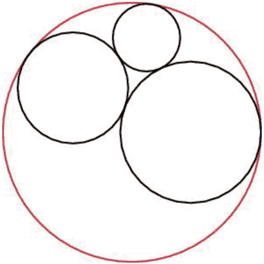

We can only have at most one "inverted" circle!

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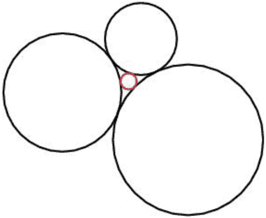

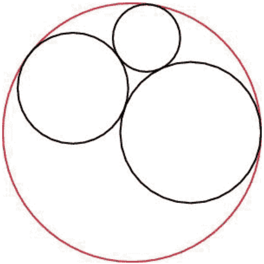

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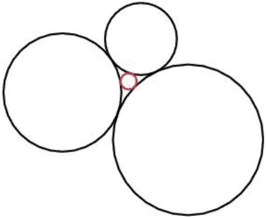

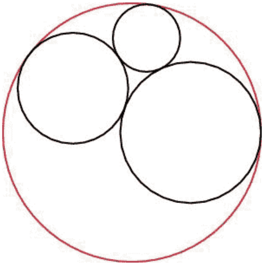

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Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

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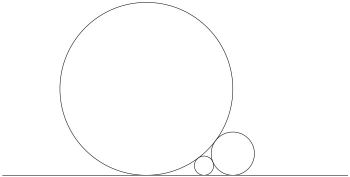

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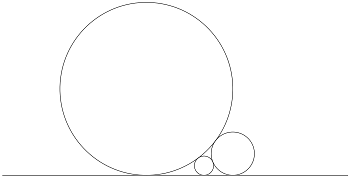

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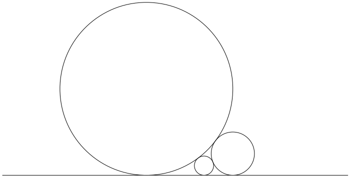

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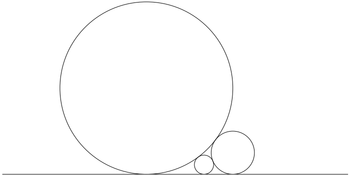

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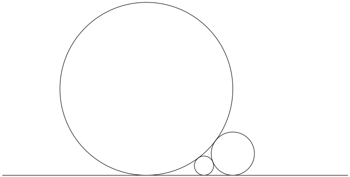

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If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

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Moreover, $d + d' = 2(a + b + c)$.

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The quadratic formula gives

$$\begin{aligned} d &= (a+b+c) \\ &\quad \pm \frac{\sqrt{4(a+b+c)^2-4\left(a^2+b^2+c^2-2ab-2bc-2ac\right)}}{2} \\ &= a+b+c\pm 2\sqrt{ab+bc+ca}. \end{aligned}$$

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Thus, there are two options for d. Their sum is $2(a+b+c)$. □

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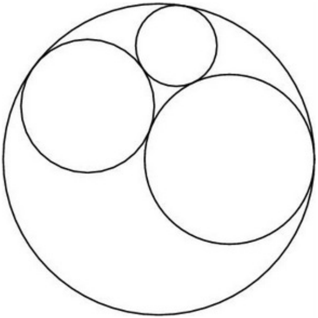

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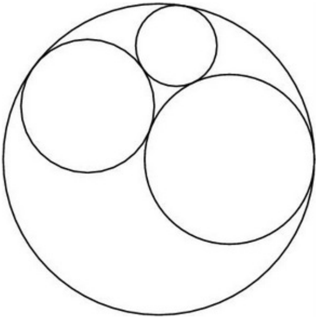
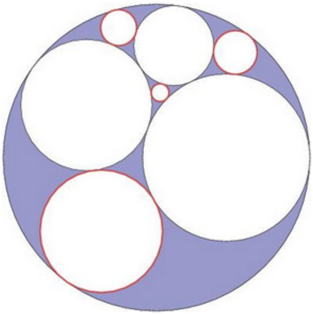
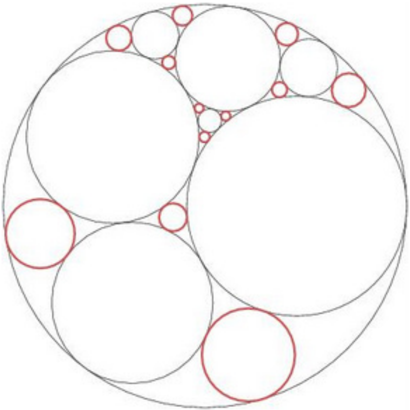




Apollonian Circle Packings



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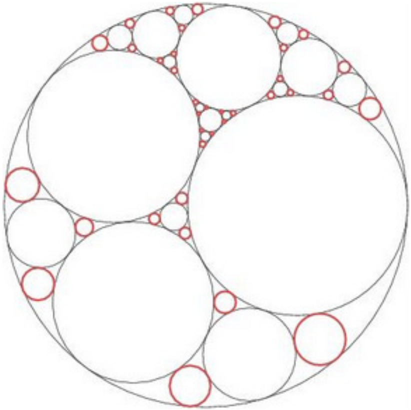

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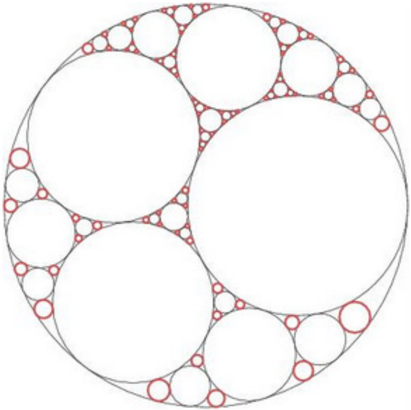

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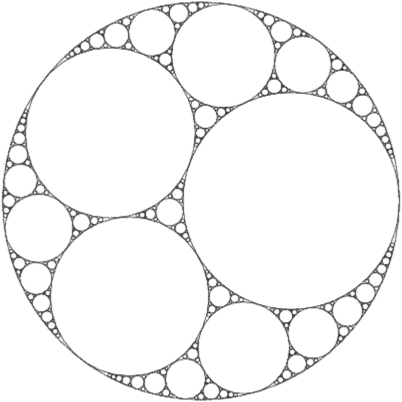

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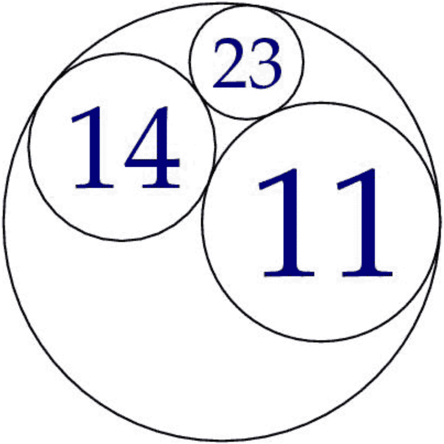
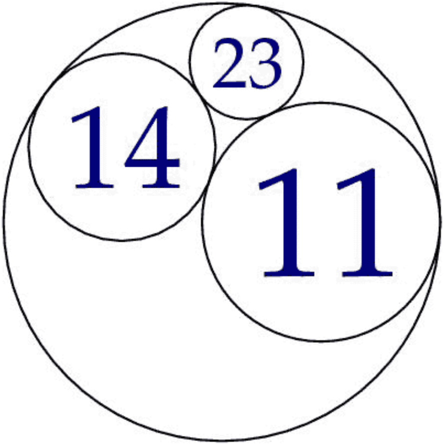


[1] Images from: AMS "When Kissing Involves Trigonometry"



$[-6, 11, 14, 23]$ [1]

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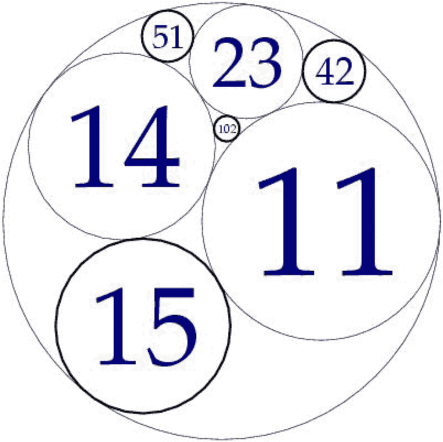

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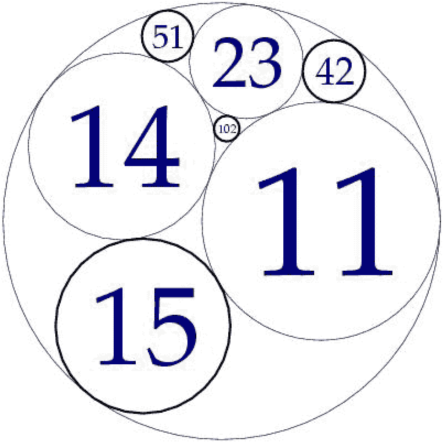

$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

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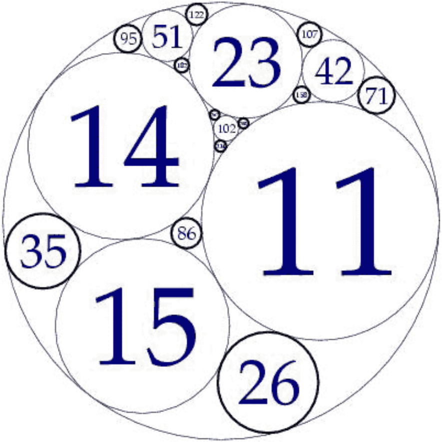

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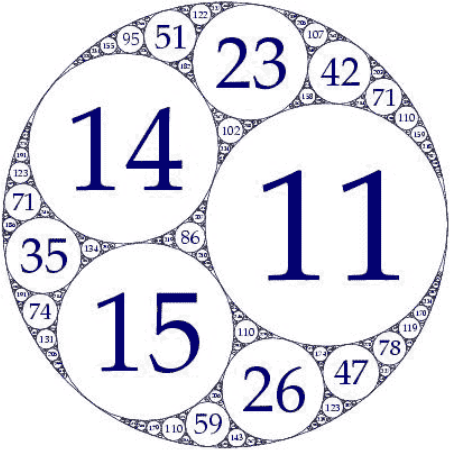

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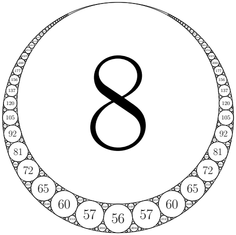

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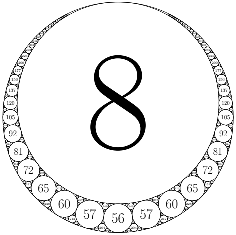

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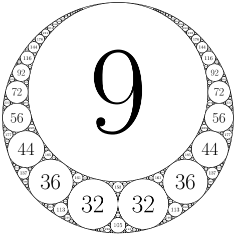

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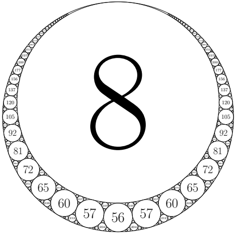

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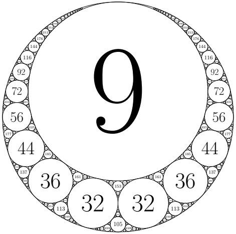

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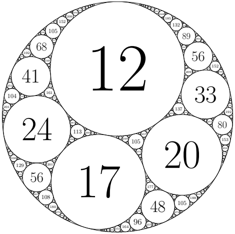

$[-7, 12, 17, 20]$.

Symmetric Packings

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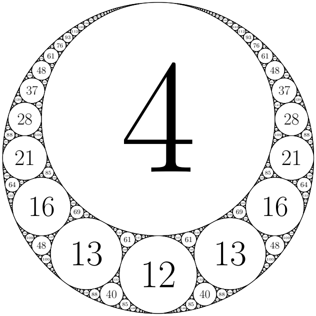

$[-3, 4, 12, 13]$

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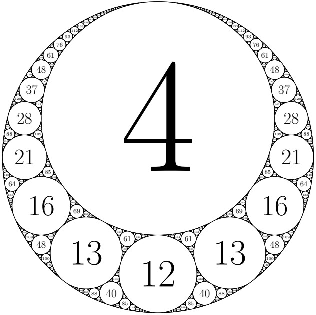

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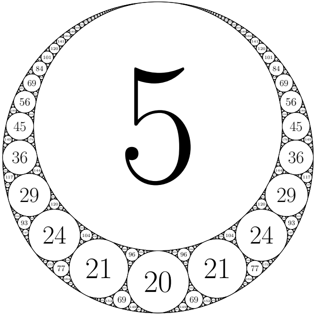

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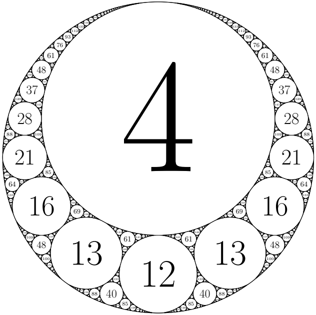

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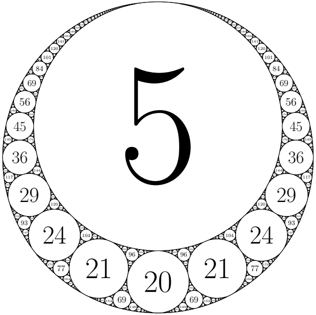

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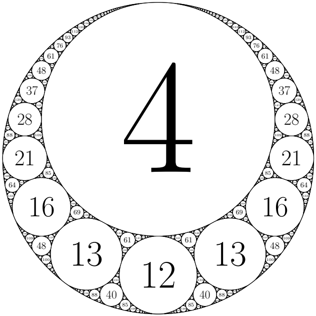

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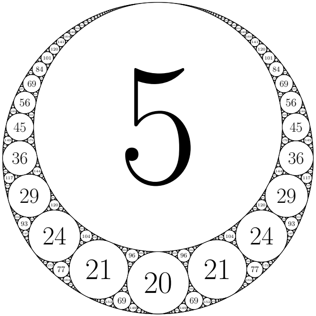

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A *sum-symmetric*

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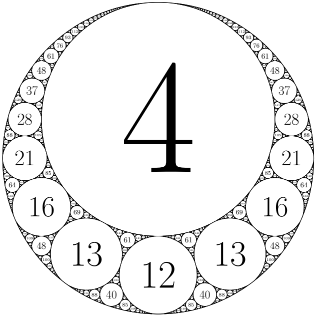

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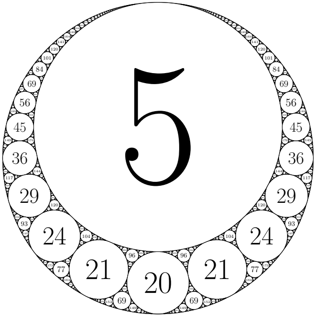

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A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a+b+c)-d=d$.

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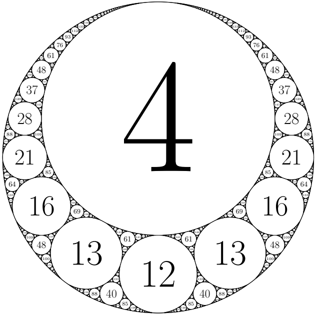

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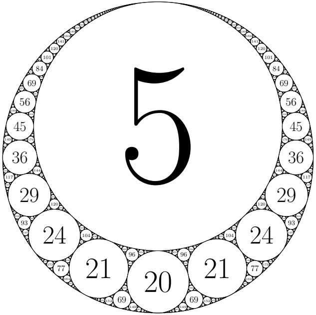

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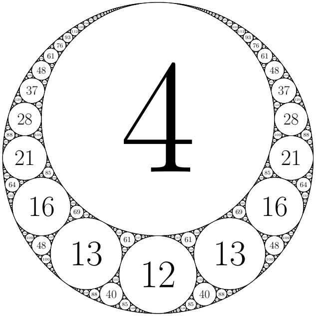

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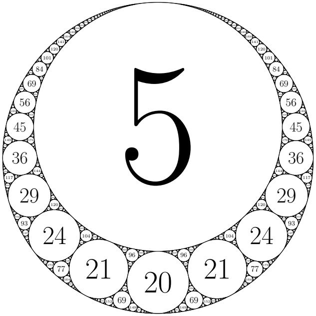

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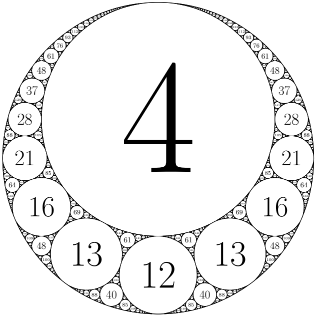

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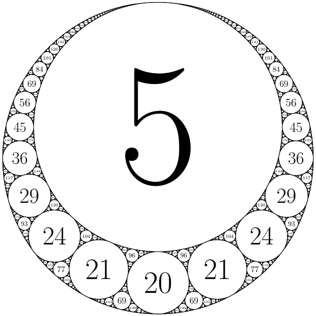

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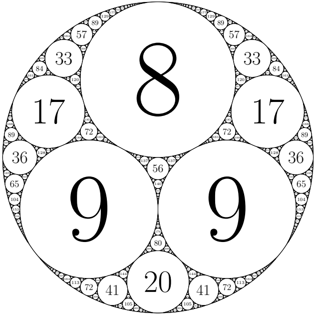

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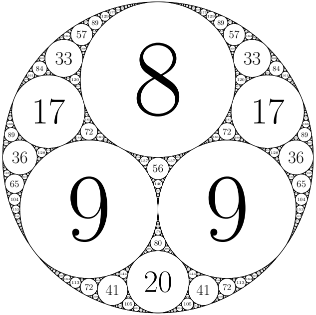
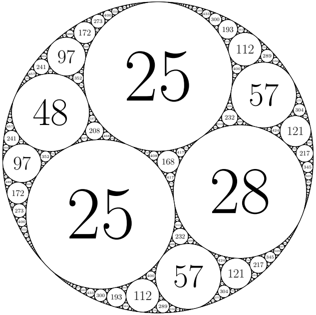
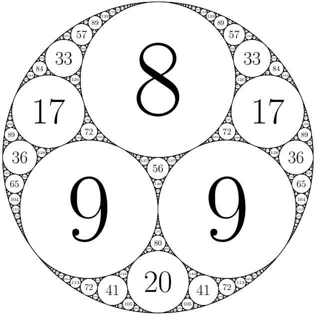


$[-4, 8, 9, 9]$

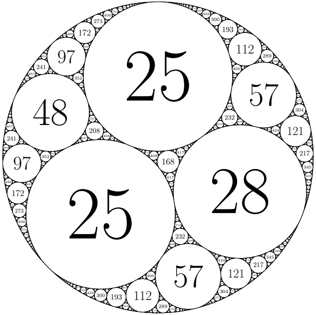


$[-12, 25, 25, 28]$

Symmetric Packings



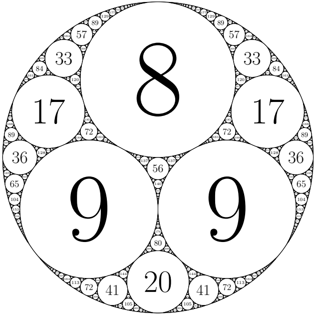
[−4, 8, 9, 9]



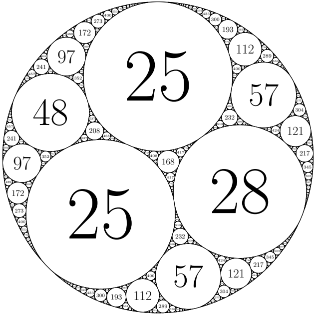
[−12, 25, 25, 28]

Definition

Symmetric Packings



[−4, 8, 9, 9]

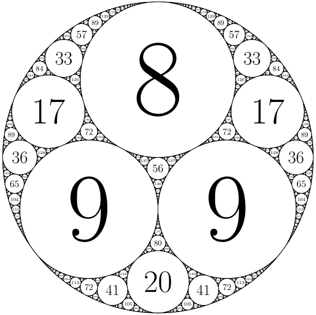


[−12, 25, 25, 28]

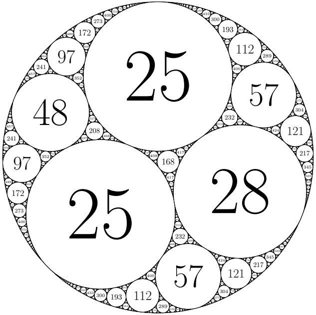
Definition

A *twin-symmetric* quadruple

Symmetric Packings



$[-4, 8, 9, 9]$



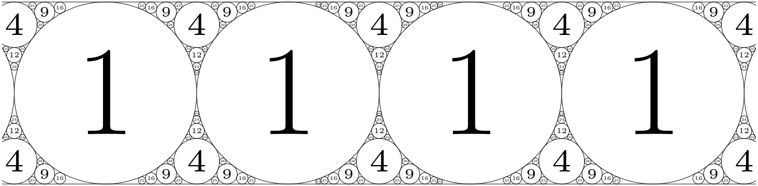
$[-12, 25, 25, 28]$

Definition

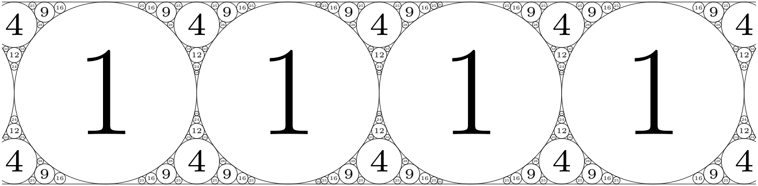
A *twin-symmetric* quadruple is a primitive reduced Descartes quadruple with $c = d$ or $c = b$.

The Two Unusual Symmetric Packings

The Two Unusual Symmetric Packings



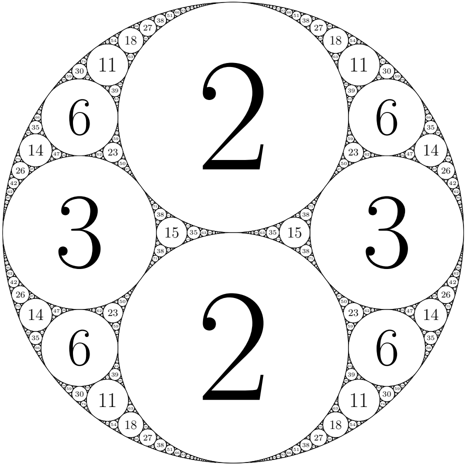
The Two Unusual Symmetric Packings



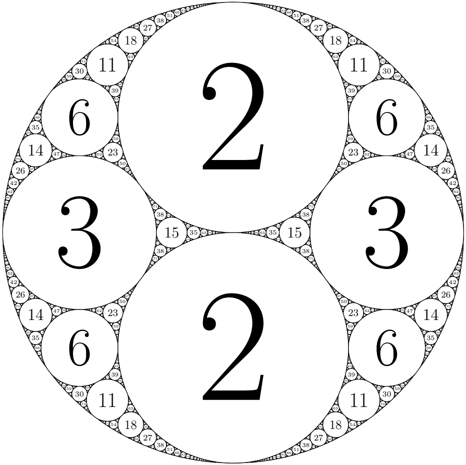
The strip packing: $[0, 0, 1, 1]$

The Two Unusual Symmetric Packings

The Two Unusual Symmetric Packings



The Two Unusual Symmetric Packings



The bug-eye packing: $[-1, 2, 2, 3]$

Symmetric Packings

Proposition

Symmetric Packings

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Symmetric Packings

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition

Symmetric Packings

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition

Only the strip and bug-eye packing are both sum-symmetric and twin-symmetric.

Sum-Symmetric Packings

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$			

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49

Sum-Symmetric Packings

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$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121

Sum-Symmetric Packings

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$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
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Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81
$[-21, 30, 70, 79]$	9	49	100

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$b+a$	$d-b$	$c+a$	$d+a$
$[-6, 10, 15, 19]$	2^2		3^2		5^2
$[-12, 21, 28, 37]$	3^2		4^2		7^2
$[-18, 22, 99, 103]$	2^2		9^2		11^2
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Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
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$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
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Given the factorization of a, we can find the entire quadruple!

Sum-Symmetric Packings

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$$\left[\underbrace{-(2\cdot 3)}_{-6},\ \underbrace{2^2+2\cdot 3}_{10},\ \underbrace{3^2+2\cdot 3}_{15},\ \underbrace{(2+3)^2-2\cdot 3}_{19}\right]$$

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$$\left[-xy,\ x^2+xy,\ y^2+xy,\ (x+y)^2-xy\right]$$

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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$$

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Try with $12 = 6 \cdot 2$:

Sum-Symmetric Packings

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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

Sum-Symmetric Packings

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$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

Sum-Symmetric Packings

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Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2\cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2\cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
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$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13]$$

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Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13] \quad (x = 3, y = 1)$$

Sum-Symmetric Packings

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Theorem

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A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

Sum-Symmetric Packings

Theorem

A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

with $\gcd(x, y) = 1$*, and* $x, y \geqslant 0$.

The Number of Sum-Symmetric Packings

The Number of Sum-Symmetric Packings

Corollary

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Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

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Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$.

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The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy but we divide by two to account for symmetry. Thus, n has $2^k/2 = 2^{k-1} = 2^{\omega(n)-1}$ sum-symmetric packings. □

Sum-Symmetric packings of 60

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Write $60 = 2^2 \cdot 3 \cdot 5$,

Sum-Symmetric packings of 60

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Sum-Symmetric packings of 60

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Sum-Symmetric packings of 60

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Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$. They are

$$(1, 60) \implies [-60, 61, 3660, 3661]$$
$$(4, 15) \implies [-60, 76, 285, 301]$$
$$(3, 20) \implies [-60, 69, 460, 469]$$
$$(5, 12) \implies [-60, 85, 204, 229]$$

Twin-Symmetric Packings

Twin-Symmetric Packings

Packings where one of the numbers is the same:

Twin-Symmetric Packings

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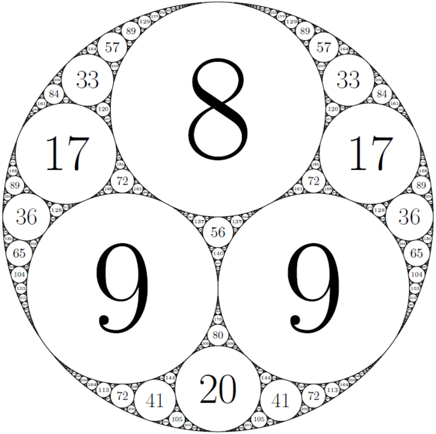

$[-4, 8, 9, 9]$

Twin-Symmetric Packings

Twin-Symmetric Packings

-2	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
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-13	$[-13, 15, 98, 98]$
-14	none
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Over the summer:

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Theorem

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All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

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Not ideal, not in terms of factorization.

Twin-Symmetric Packings

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Improved to:

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A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{(x+y)^2}{2},\ \frac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x > y \end{cases}$$

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with $\gcd(x,y)=1$.

Further improved to:

Twin-Symmetric Packings

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A twin-symmetric quadruple is one of following two forms

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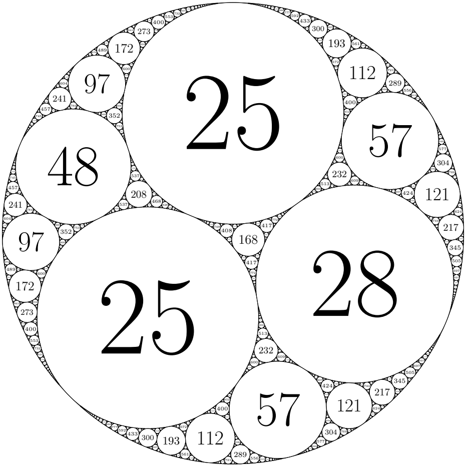
$$[-3, 3+2(3)^2, 5^2, 5^2] \implies [-12, 48, 25, 25]$$

Twin-symmetric Packings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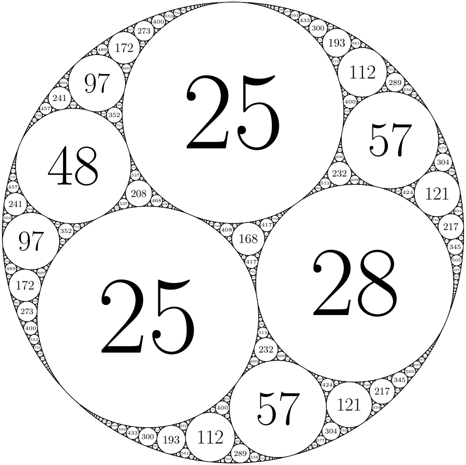
$[-12, 48, 25, 25]$



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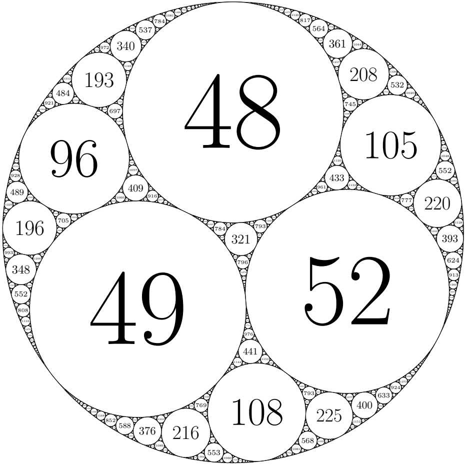
Corollary

A natural number n has $(1 - \delta_n) \cdot 2^{\omega(n)-1}$ twin-symmetric packings where $\omega(n)$ is the number of distinct prime divisors of n.

Non-symmetric Packings

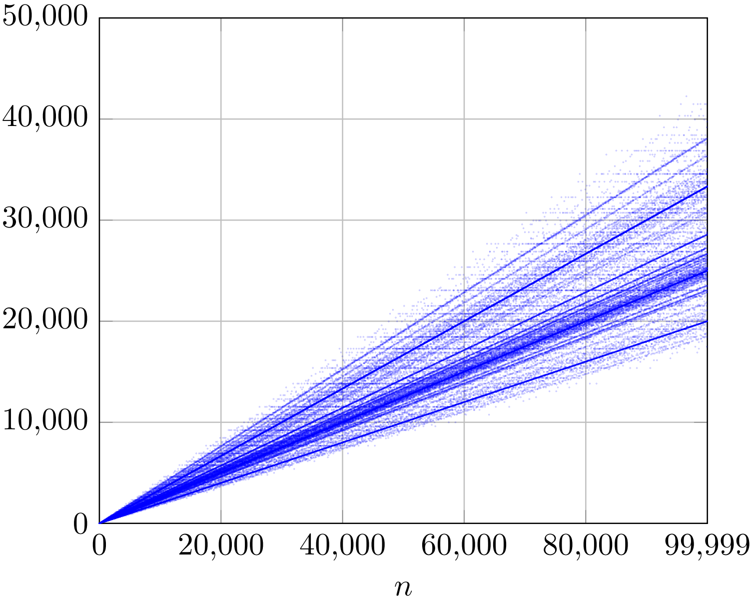
Non-symmetric Packings



$[-23, 48, 49, 52]$.

Non-symmetric Packings

Non-symmetric Packings



Families of non-symmetric packings

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Every packing can be written

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Every packing can be written

$$\left[-n,\ n+k,\ \frac{n^2+kn+\alpha^2}{k},\ \frac{n^2+kn+(k-\alpha)^2}{k}\right]$$

(Bridges, Tai, and Koziol.)

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where $\chi_{-4}(n)=(-1)^{(n-1)/2}$ for n odd and 0 for n even. (Due to Graham, Lagarias, Mallows, Wilks, Yan)

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$$\frac{n}{4}\prod_{p|n}\left(1-\frac{\chi_{-4}(p)}{p}\right)+2^{\omega(n)-\delta_n-1}-\underbrace{(1-\delta_n)\cdot 2^{\omega(n)-1}}_{\text{twin-symmetric}}-\underbrace{2^{\omega(n)-1}}_{\text{sum-symmetric}}$$

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□

Extended Example

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Sum-symmetric: $\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$

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Total number is

$$\frac{20}{4} \prod_{p \mid 20} \left(1 - \frac{\chi_{-4}(p)}{p}\right) + 2^{\omega(20) - \delta_{20} - 1}$$

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$\Longrightarrow$ non-symmetric: $6-2-2=2$.

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$\implies$ non-symmetric: $6 - 2 - 2 = 2.$

Coprime factor pairs of 20: $(1, 20)$ and $(4, 5)$.

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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$$

$(1,20)$

Extended Example - sum-symmetric

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$$

$$(1,20) \implies \left[-1\cdot 20,\ 1(1+20),\ 20(1+20),\ (1+20)^2-1\cdot 20\right]$$

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$$= [-20,\ 21,\ 420,\ 421]$$

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$$(4,5)$$

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Clyde Kertzer

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$$= [-20,\ 21,\ 420,\ 421]$$

$$(4,5) \implies \left[-4\cdot 5,\ 4(4+5),\ 5(4+5),\ (4+5)^2-4\cdot 5\right]$$

$$= [-20,\ 36,\ 45,\ 61]$$

Extended Example - twin-symmetric

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \qquad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \qquad x > y \end{cases}$$

$(1, 10)$

Extended Example - twin-symmetric

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$(2, 5)$

Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

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Extended Example - twin-symmetric

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \qquad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \qquad x > y \end{cases}$$

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Extended Example

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[−20, 21, 420, 421]

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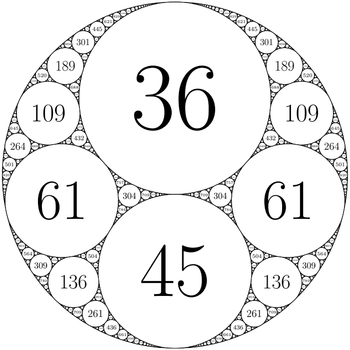

[−20, 36, 45, 61]

Extended Example

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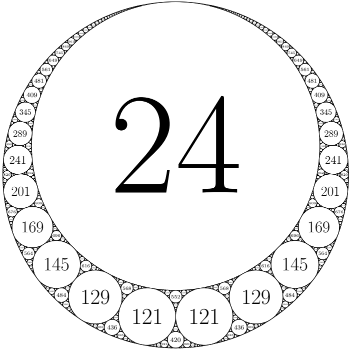
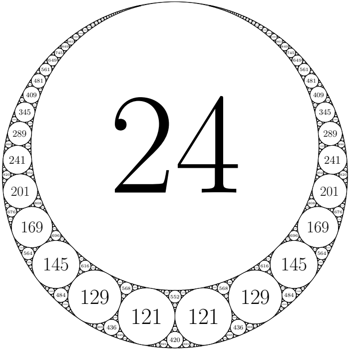


$[-20,\ 24,\ 121,\ 121]$

Extended Example



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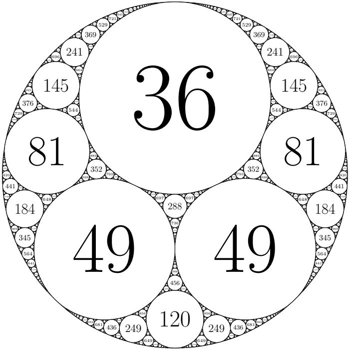

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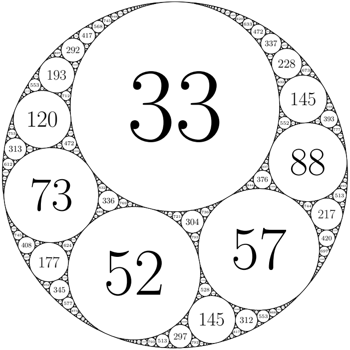

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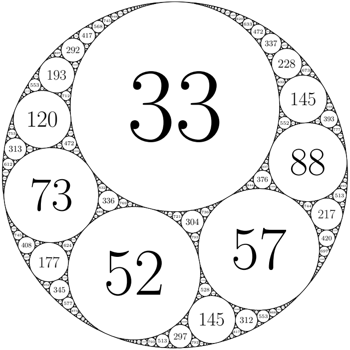

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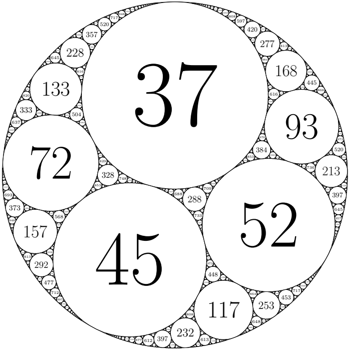


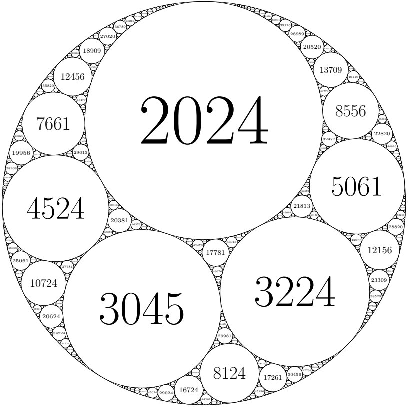
[−20, 37, 45, 52]

Acknowledgments

I am incredibly grateful to Professor Katherine Stange and Dr. James Rickards for taking me under their wing over the previous summer's REU. Their mentorship and encouragement inspired me to pursue not only this honors thesis, but a math conference across the country. Under their guidance, I have learned just how fun math research can be! Working on this thesis has been one of the most fulfilling projects I have undertaken.

I am also thankful to the Honors committee for reviewing my thesis. Without them, the honors program would not be possible.

Thank You!



Images generated using James Rickards' Code.

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Proof of Descartes Equation

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First, we need a trigonometric lemma

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$$\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1 + 2\cos\alpha\cos\beta\cos\theta.$$

Proof of the Lemma

Proof of the Lemma

Proof.

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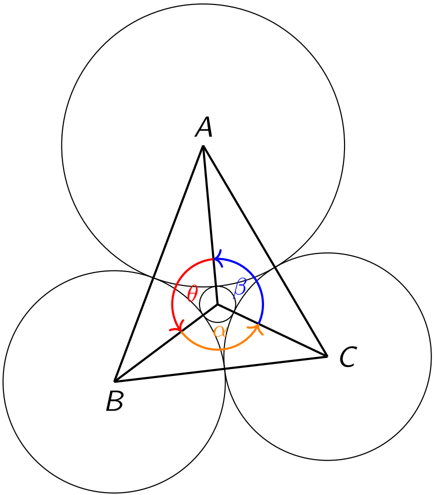
$$\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\theta &= \\
&= \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\theta}{2} \\
&= \frac{3}{2} + \frac{\cos 2\alpha + \cos 2\beta}{2} + \frac{\cos(2\pi - (2\alpha + 2\beta))}{2} \\
&= \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{\cos 2(\alpha+\beta)}{2} \\
&= \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{2\cos^2(\alpha+\beta) - 1}{2} \\
&= 1 + \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos^2(\alpha+\beta) \\
&= 1 + (\cos(\alpha-\beta) + \cos(\alpha+\beta))\cos(2\pi-\theta) \\
&= 1 + 2\cos\alpha\cos\beta\cos\theta.
\end{aligned}$$

□

Proof of the Lemma



Four mutually tangent circles with centers A, B, C, and D.

Proof of the Descartes Equation

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Proof.

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$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad AC = r_A + r_C$$

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Let $\angle BDC = \alpha$, $\angle CDA = \beta$, and $\angle ADB = \theta$.

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Let $\angle BDC = \alpha$, $\angle CDA = \beta$, and $\angle ADB = \theta$. The law of cosines in $\triangle ADB$ yields

$$\begin{aligned}\cos\theta &= \frac{AD^2 + BD^2 - AB^2}{2 \cdot AD \cdot BD} \\ &= \frac{(r_A + r_D)^2 + (r_B + r_D)^2 - (r_A + r_B)^2}{2(r_A + r_D)(r_B + r_D)} \\ &= \frac{2r_D^2 + 2r_D(r_A + r_B) - 2r_A r_B}{2(r_A + r_D)(r_B + r_D)} \\ &= 1 - \frac{2r_A r_B}{(r_A + r_D)(r_B + r_D)}.\end{aligned}$$

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$$\cos\alpha = 1 - \frac{2k_D^2}{(k_B + k_D)(k_C + k_D)} = 1 - \lambda_\alpha$$

$$\cos\beta = 1 - \frac{2k_D^2}{(k_A + k_D)(k_C + k_D)} = 1 - \lambda_\beta$$

$$\cos\theta = 1 - \frac{2k_D^2}{(k_A + k_D)(k_B + k_D)} = 1 - \lambda_\theta.$$

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$$(1-\lambda_\alpha)^2 + (1-\lambda_\beta)^2 + (1-\lambda_\theta)^2 = 1 + 2(1-\lambda_\alpha)(1-\lambda_\beta)(1-\lambda_\theta)$$

$$\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\theta^2 + 2\lambda_\alpha\lambda_\beta\lambda_\theta = 2(\lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\theta + \lambda_\alpha\lambda_\theta)$$

$$\frac{\lambda_\alpha}{\lambda_\beta\lambda_\theta} + \frac{\lambda_\beta}{\lambda_\alpha\lambda_\theta} + \frac{\lambda_\theta}{\lambda_\alpha\lambda_\beta} + 2 = 2\left(\frac{1}{\lambda_\alpha} + \frac{1}{\lambda_\beta} + \frac{1}{\lambda_\theta}\right).$$

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$$\frac{\lambda_\alpha}{\lambda_\beta\lambda_\theta}+\frac{\lambda_\beta}{\lambda_\alpha\lambda_\theta}+\frac{\lambda_\theta}{\lambda_\alpha\lambda_\beta}+2=2\left(\frac{1}{\lambda_\alpha}+\frac{1}{\lambda_\beta}+\frac{1}{\lambda_\theta}\right).$$

Substituting back our values for the λs we find

$$\frac{(k_A+k_D)^2}{2k_D^2}+\frac{(k_B+k_D)^2}{2k_D^2}+\frac{(k_C+k_D)^2}{2k_D^2}+2=$$
$$2\frac{(k_B+k_D)(k_C+k_D)}{2k_D^2}+2\frac{(k_A+k_D)(k_B+k_D)}{2k_D^2}$$
$$+2\frac{(k_A+k_D)(k_B+k_D)}{2k_D^2}.$$

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$$\begin{aligned} k_A^2 + k_B^2 + k_C^2 + 2k_D(k_A + k_B + k_C) + 7k_D^2 \\ = 6k_D^2 + 4k_D(k_A + k_B + k_C) \\ + 2(k_A k_B + k_B k_C + k_A k_C) \end{aligned}$$

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$$\begin{aligned} k_A^2 + k_B^2 + k_C^2 + k_D^2 &= 2k_D(k_A + k_B + k_C) \\ &\quad + 2(k_A k_B + k_B k_C + k_A k_C) \\ &= (k_A + k_B + k_C + k_D)^2 \\ &\quad - (k_A^2 + k_B^2 + k_C^2 + k_D^2) \\ 2(k_A^2 + k_B^2 + k_C^2 + k_D^2) &= (k_A + k_B + k_C + k_D)^2. \end{aligned}$$

□

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(i) $a + b = d - c$

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Proving the Parameterization

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$$\begin{aligned}(a+b+c+d)^2 &= 2\left(a^2+b^2+c^2+d^2\right)\\(d-c+c+d)^2 &= 2a^2+2b^2+2c^2+2d^2\\4d^2 &= 2\left(a^2+b^2+c^2\right)+2d^2\\d^2 &= a^2+b^2+c^2.\end{aligned}$$

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(iii) Use substitutions from parts (i) and (ii) to find

$$\begin{aligned}a+b+c &= d\\ (a+b+c)^2 &= a^2+b^2+c^2\\ a^2+b^2+c^2+2ab+2ac+2bc &= a^2+b^2+c^2\\ ab+ac+bc &= 0\end{aligned}$$

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Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$.

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Let $g = \gcd(a + b, a + c)$

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Let $g = \gcd(a + b, a + c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y.

Proving the Parameterization

Proof.

Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

$$ab + ac + bc = 0$$
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Let $g = \gcd(a + b, a + c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y. This yields $gxy = -a$.

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Thus, we have that

$$\begin{aligned} a &= -gxy \\ b &= gx(x + y) \\ c &= gy(x + y) \\ d &= g((x + y)^2 - xy). \end{aligned}$$

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$$
\begin{aligned}
a &= -xy \\
b &= x(x+y) \\
c &= y(x+y) \\
d &= (x+y)^2 - xy.
\end{aligned}
$$

with $\gcd(x, y) = 1$. □