The Local-Global Conjecture is False

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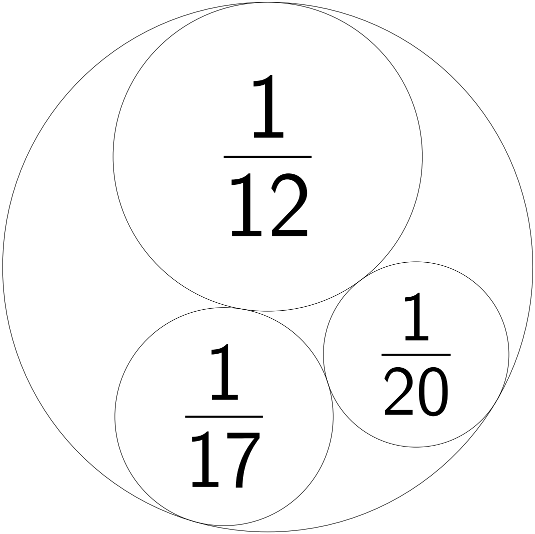
Circle Packings



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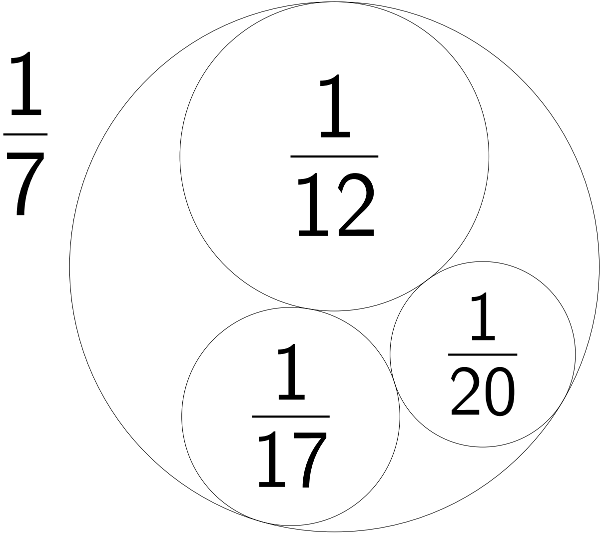

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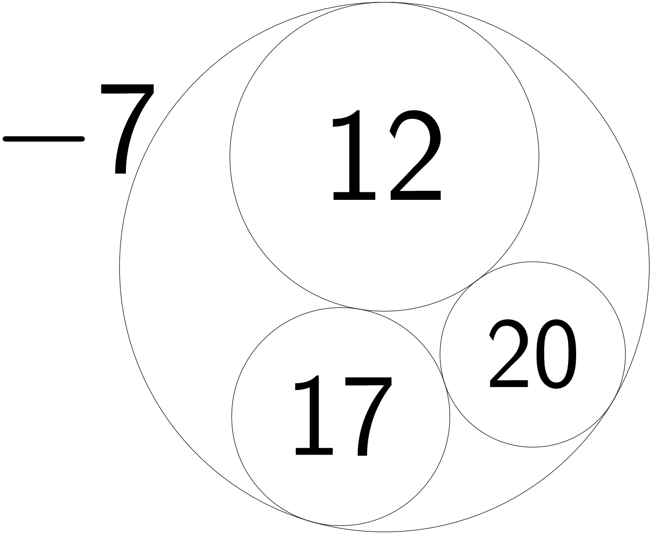

Summary: The Descartes Equation

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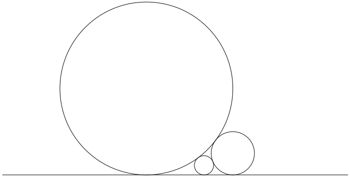

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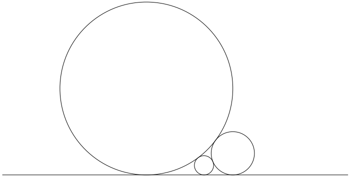

Circle with infinite radius (Curvature 0)

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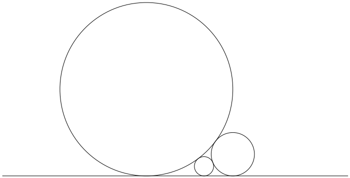

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Four mutually tangent circles are called a *Descartes Quadruple*.

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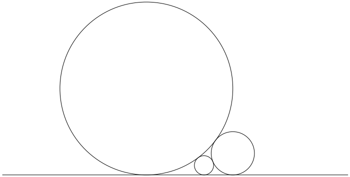

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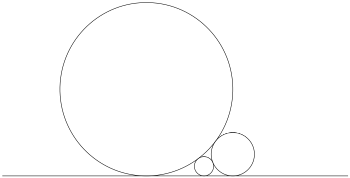

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Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

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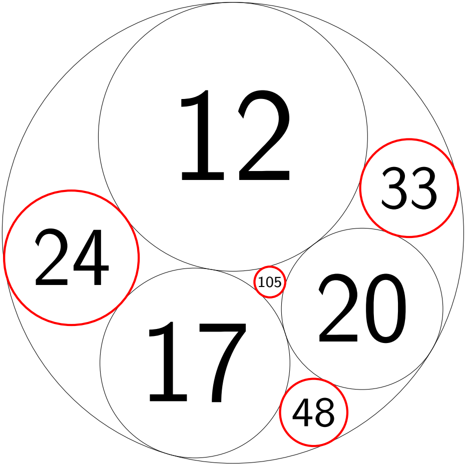

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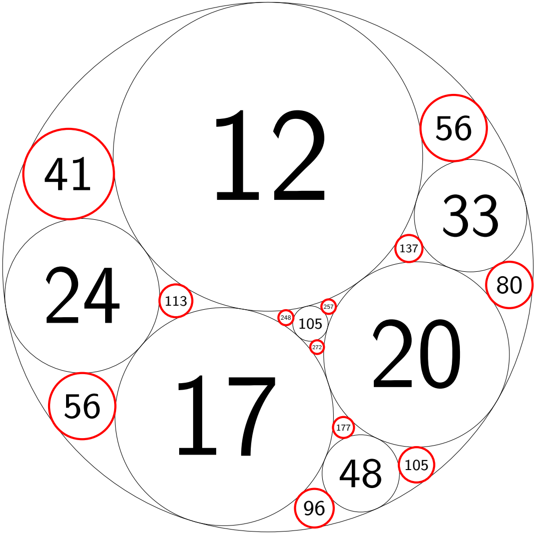

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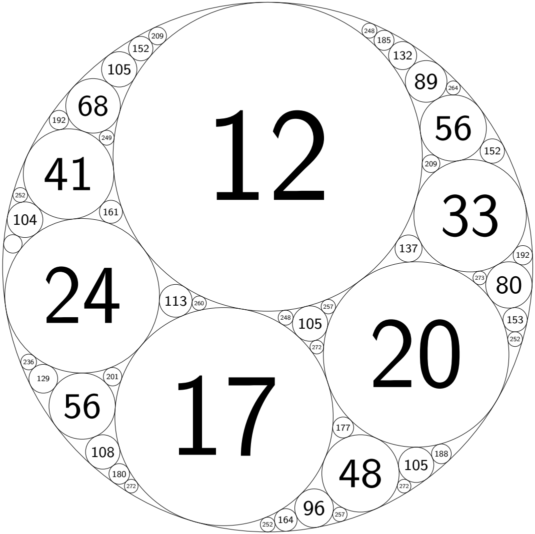

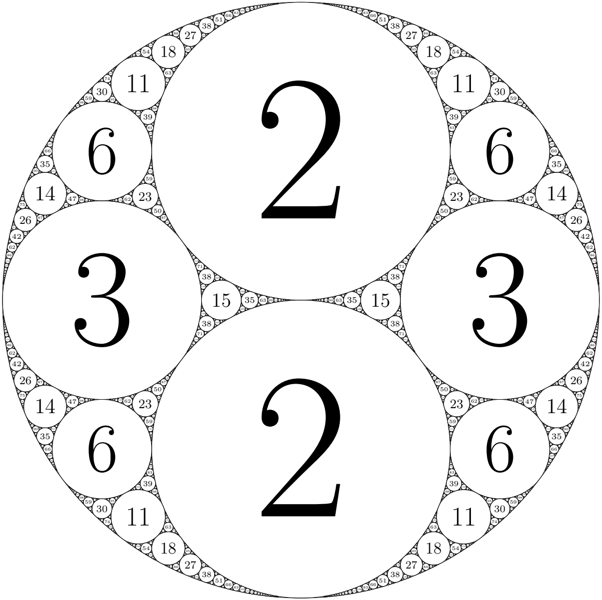
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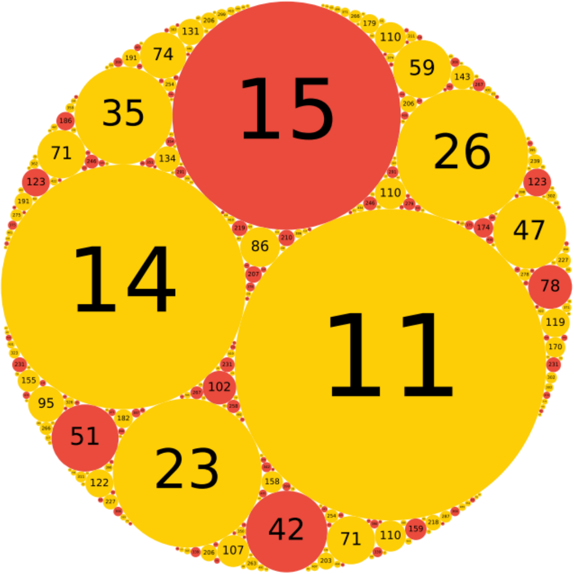

Allowed Residues

Theorem (Fuchs-Sanden (2010))

If a congruence obstruction appears, then it appears modulo 24.

Type	Allowed Residues
$(6,1)$	$0,1,4,9,12,16$
$(6,5)$	$0,5,8,12,20,21$
$(6,13)$	$0,4,12,13,16,21$
$(6,17)$	$0,8,9,12,17,20$
$(8,7)$	$3,6,7,10,15,18,19,22$
$(8,11)$	$2,3,6,11,14,15,18,23$

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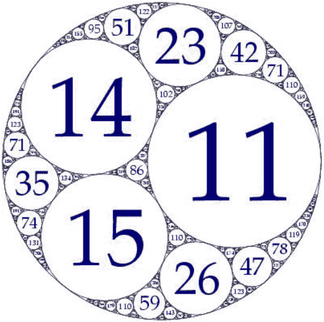

$[-6, 11, 14, 15]$

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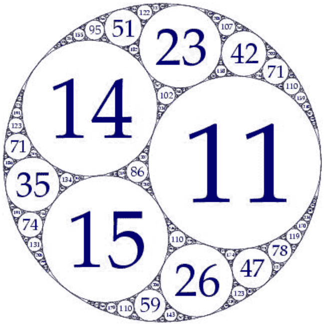

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No room for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, . . .

Missing Curvatures?

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } [-1, 2, 2, 3]$$

$$5 \cdot 10^8 \text{ for } [-11, 21, 24, 28]$$

and observed for $[-11, 21, 24, 28]$, there were still a small number (up to 0.013%) of missing curvatures in the range $\left(4 \cdot 10^8, 5 \cdot 10^8\right)$ for residue classes $0, 4, 12, 16 \bmod 24$.

Local-to-global

Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden, 2010)

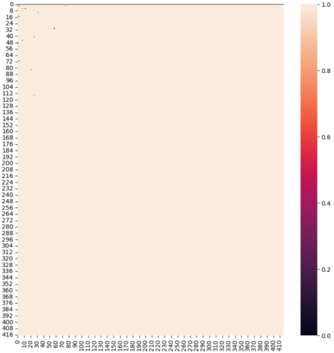
Curvatures satisfy a congruence condition mod 24, and all sufficiently large integers satisfying this condition appear.

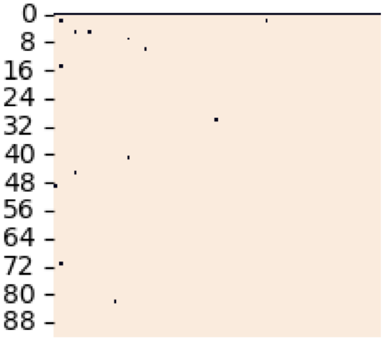
Theorem (Bourgain-Kontorovich, 2014)

The number of missing curvatures up to N is at most $O(N^{1-\varepsilon})$ for some computable $\varepsilon > 0$.

1. Fix a pair of curvatures, and study what packings contain them.
2. Plot: for an admissible pair of residue classes modulo 24, black dot if no packing has that pair.
3. Local-global: finitely many black dots on any row or column.

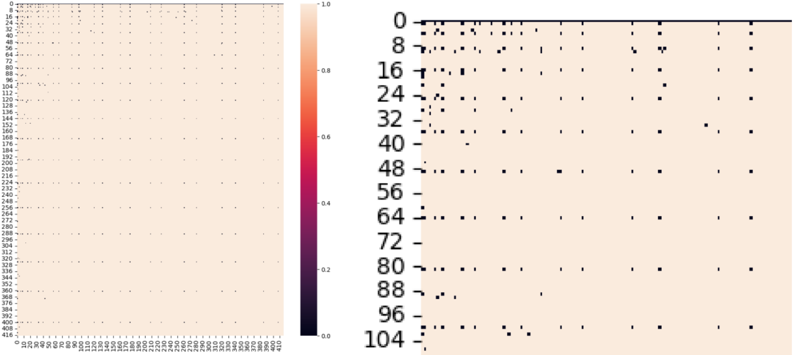
Typical graph





Residue classes 0 (mod 24) and 12 (mod 24) (Summer Haag)

One weird graph



Residue classes 0 (mod 24) and 8 (mod 24) (Summer Haag)

No bug–The conjecture is false!

Theorem (Haag-Kertzer-Rickards-Stange)

The packing $[-3, 5, 8, 8]$ *has no square curvatures.*

The New Conjecture

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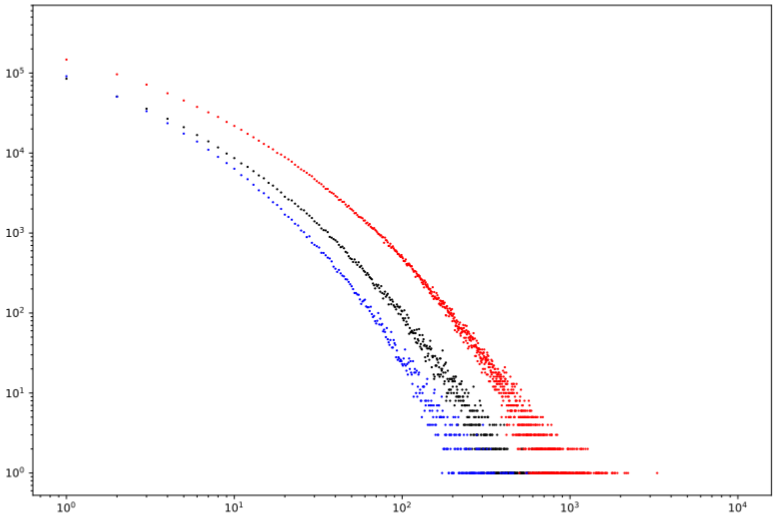
The type of a packing implies the existence of certain quadratic and quartic obstructions:

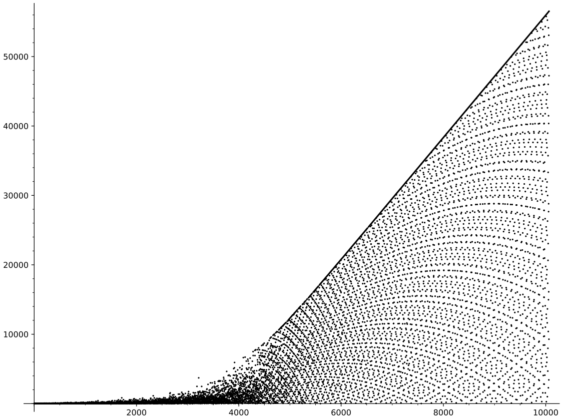
Type	n^2 Obstructions	n^4 Obstructions	L-G false	L-G open
$(6, 1, 1, -1)$		$n^4, 4n^4, 9n^4, 36n^4$	$0, 1, 4, 9, 12, 16$	
$(6, 1, -1)$	$n^2, 2n^2, 3n^2, 6n^2$		$0, 1, 4, 9, 12, 16$	
$(6, 5, 1)$	$2n^2, 3n^2$		$0, 8, 12$	$5, 20, 21$
$(6, 5, -1)$	$n^2, 6n^2$		$0, 12$	$5, 8, 20, 21$
$(6, 13, 1)$	$2n^2, 6n^2$		0	$4, 12, 13, 16, 21$
$(6, 13, -1)$	$n^2, 3n^2$		$0, 4, 12, 16$	$13, 21$
$(6, 17, 1, 1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, 1, -1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, -1)$	$n^2, 2n^2$		$0, 8, 9, 12$	$17, 20$
$(8, 7, 1)$	$3n^2, 6n^2$		$3, 6$	$7, 10, 15, 18, 19, 22$
$(8, 7, -1)$	$2n^2$		18	$3, 6, 7, 10, 15, 19, 22$
$(8, 11, -1)$	$2n^2, 3n^2, 6n^2$		$2, 3, 6, 18$	$11, 14, 15, 23$

Missing curvatures dropping off



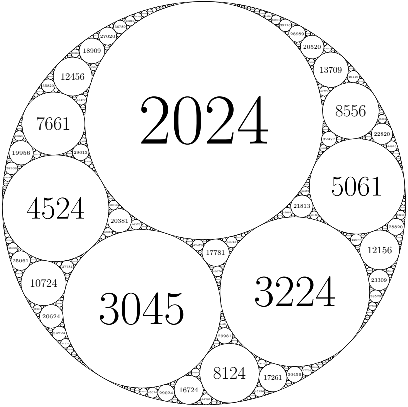
A loglog plot of the probability a curvature is missing, as curvature size increases.

Successive differences



Successive differences of missing curvatures in the packing $[-4, 5, 20, 21]$. The quadratic families $2n^2$ and $3n^2$ begin to predominate (the missing set has 3659 elements $< 10^{10}$, and occur increasingly sparsely.)

Thank You!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