Apollonian Circle Packings

University of Colorado Boulder

July 22, 2025

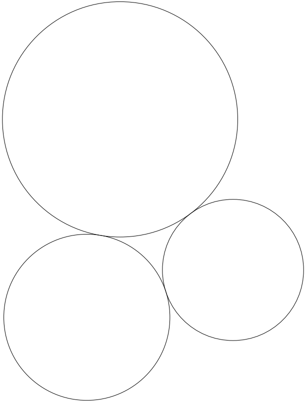
2000 years ago...

2000 years ago...

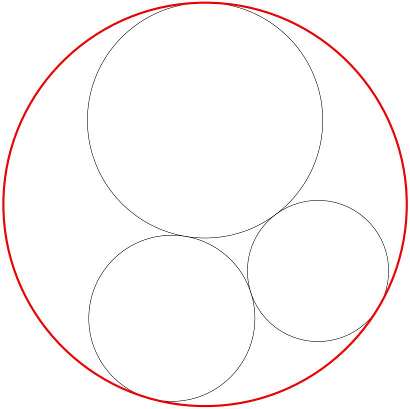
2000 years ago...

Circle Packing

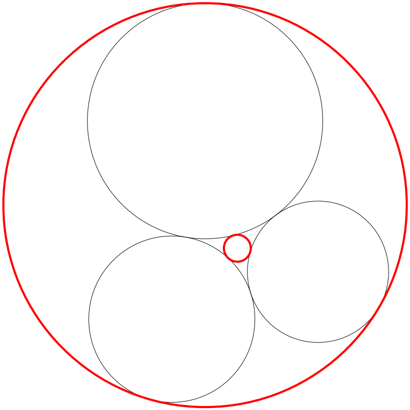
Circle Packing



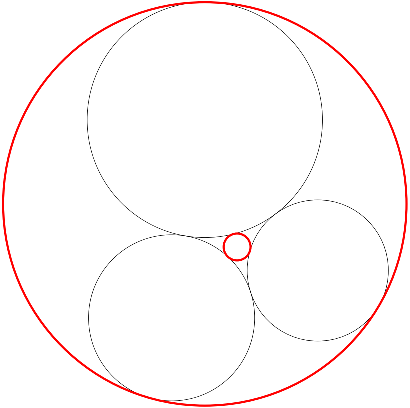
Circle Packing



Circle Packing

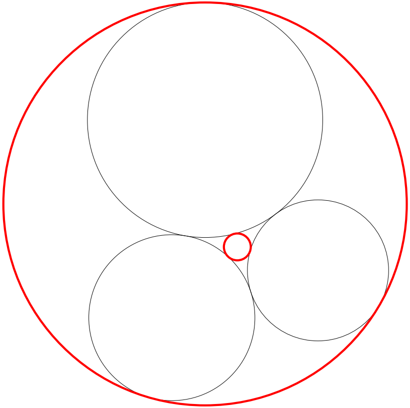


Circle Packing



Theorem (Apollonius)

Circle Packing



Theorem (Apollonius)

Given three mutually tangent circles, there are exactly two other circles tangent to all three.

To the early 1600s...

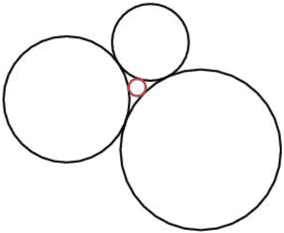
To the early 1600s...

To the early 1600s...

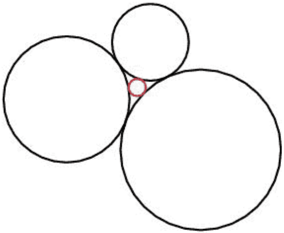
To the early 1600s...

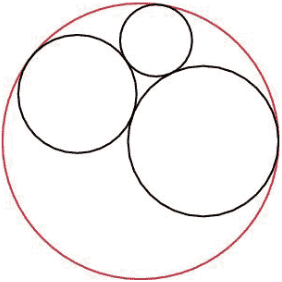
Descartes

Descartes

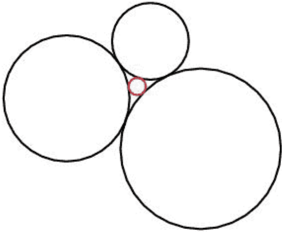


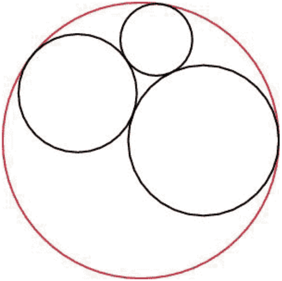
Descartes





Descartes

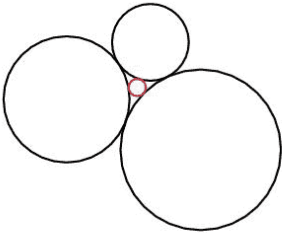


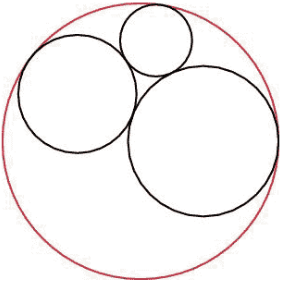


Definition

A *Descartes Quadruple* is a set of four mutually tangent circles with disjoint interiors.

Descartes



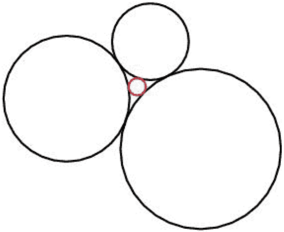


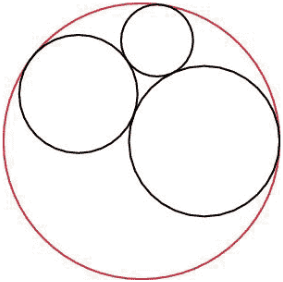
Definition

A *Descartes Quadruple* is a set of four mutually tangent circles with disjoint interiors.

Can only have at most one “inverted” circle!

Descartes





Definition

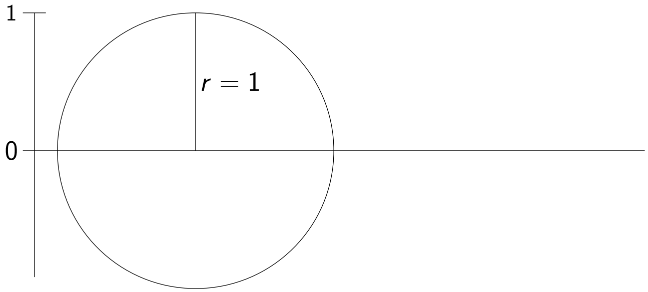
A *Descartes Quadruple* is a set of four mutually tangent circles with disjoint interiors.

Can only have at most one “inverted” circle! $\implies$ negative curvature

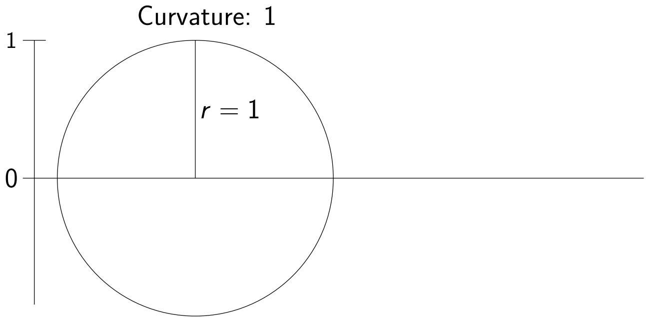
Curvature

Curvature

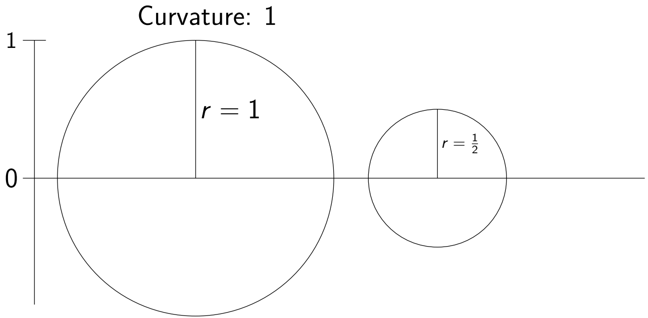
Curvature



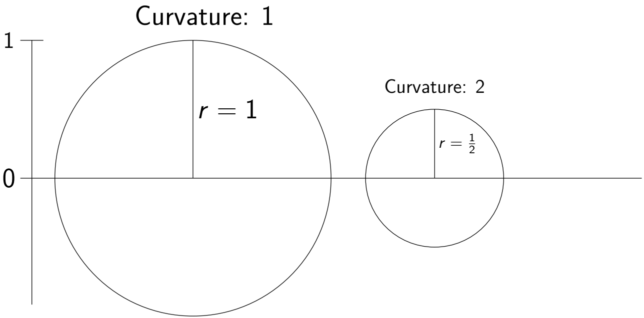
Curvature



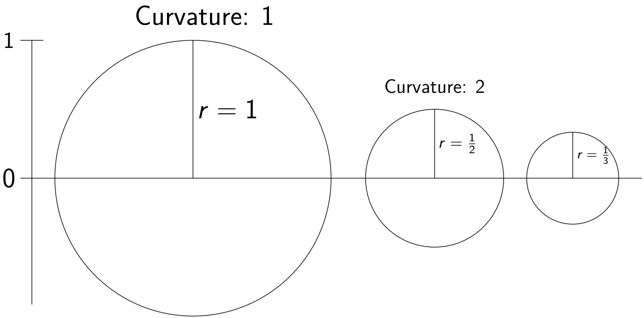
Curvature



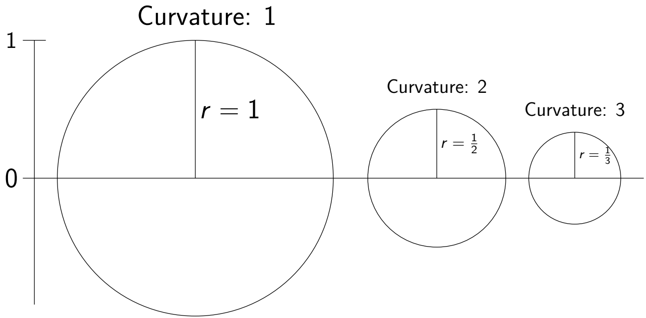
Curvature



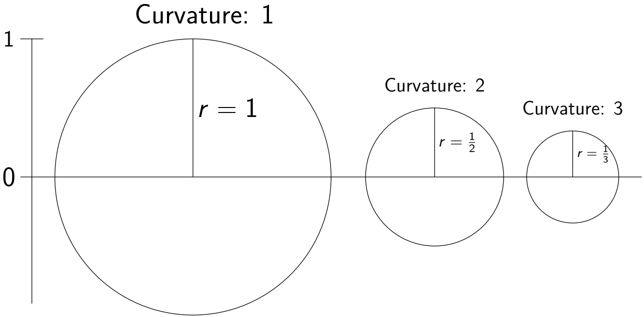
Curvature



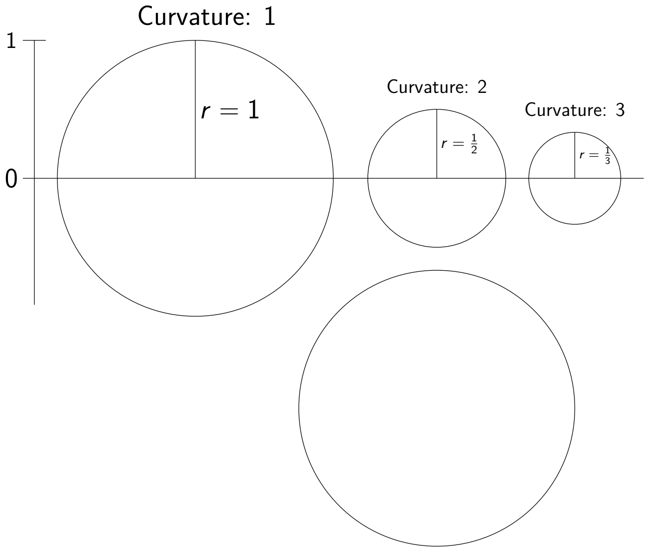
Curvature



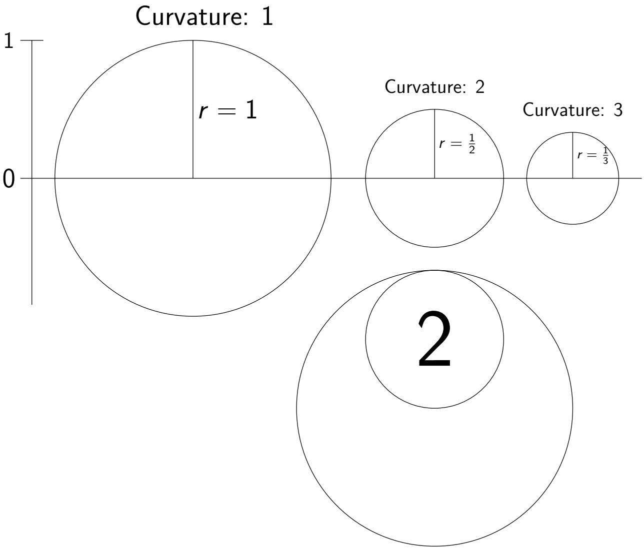
Curvature



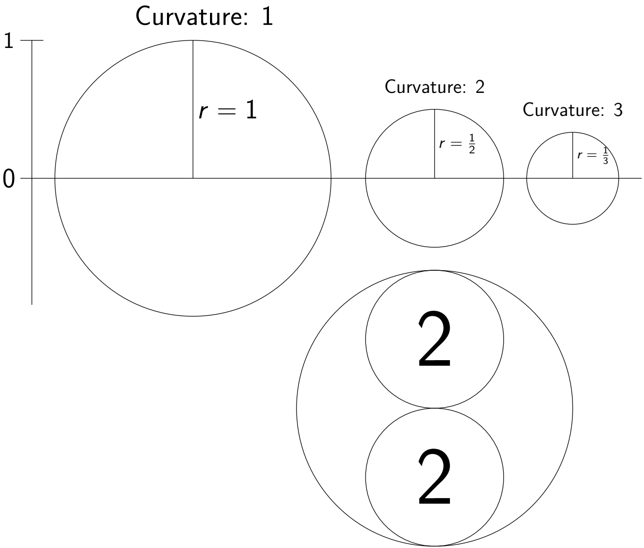
Curvature



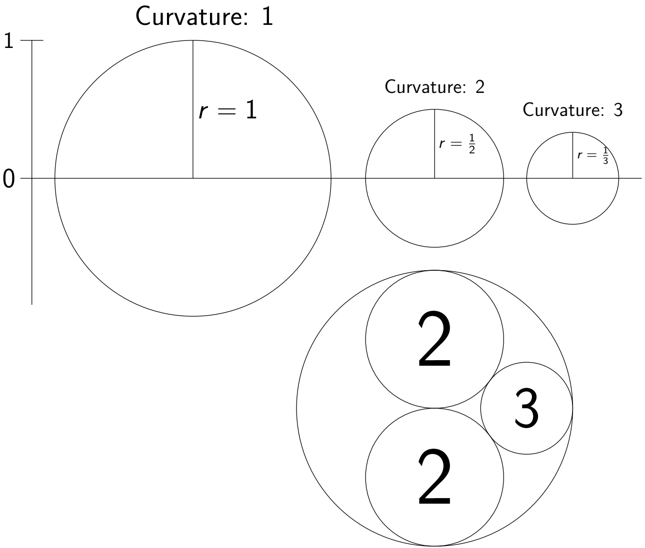
Curvature



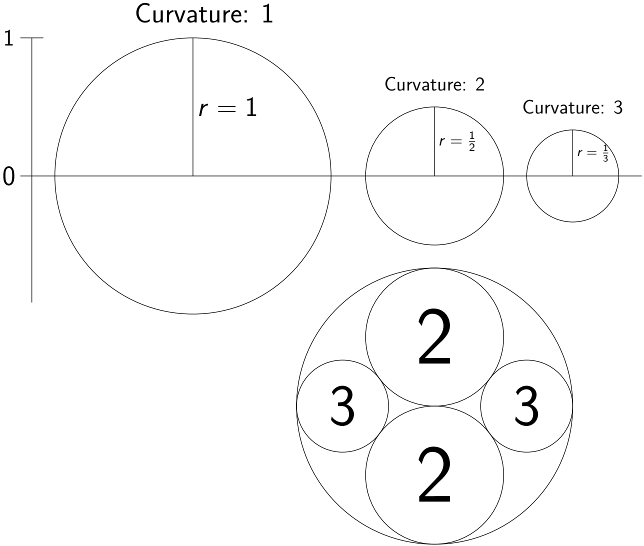
Curvature


Curvature: 1
1
0
$r = 1$
Curvature: 2
$r = \frac{1}{2}$
Curvature: 3
$r = \frac{1}{3}$
2
2

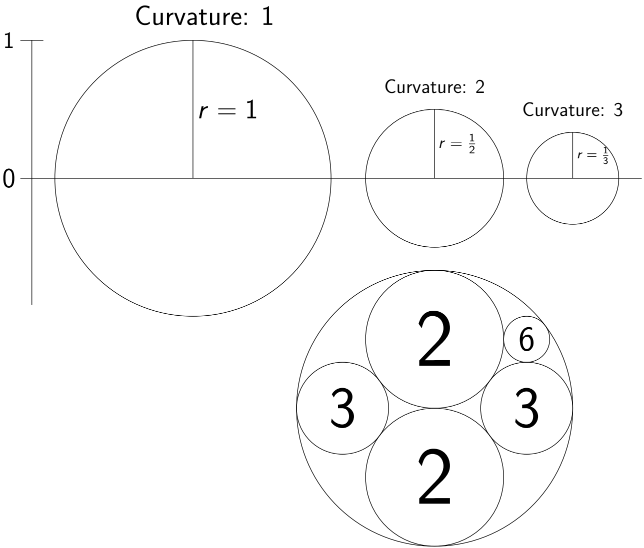
Curvature


Curvature: 1
1
0
$r = 1$
Curvature: 2
$r = \frac{1}{2}$
Curvature: 3
$r = \frac{1}{3}$
2
3
2

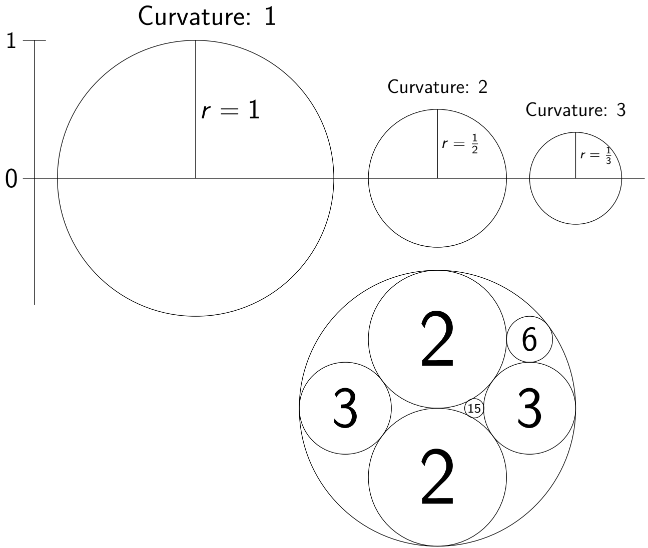
Curvature


Curvature: 1
1
0
$r = 1$
Curvature: 2
$r = \frac{1}{2}$
Curvature: 3
$r = \frac{1}{3}$
2
3
3
2

Curvature


Curvature: 1
1
0
$r = 1$
Curvature: 2
$r = \frac{1}{2}$
Curvature: 3
$r = \frac{1}{3}$
2
6
3
3
2

Curvature


Curvature: 1
1
0
$r = 1$
Curvature: 2
$r = \frac{1}{2}$
Curvature: 3
$r = \frac{1}{3}$
2
6
3
15
3
2

Descartes

Descartes

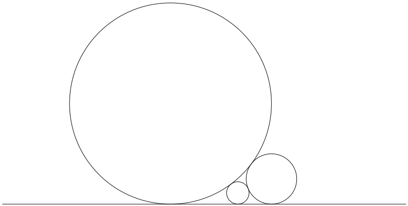
Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

Descartes

Definition

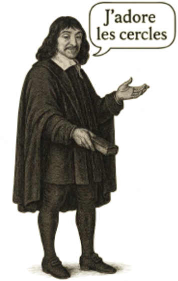
The *curvature* of a circle with radius r is defined to be $1/r$.

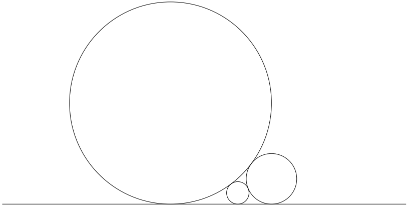


Descartes

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

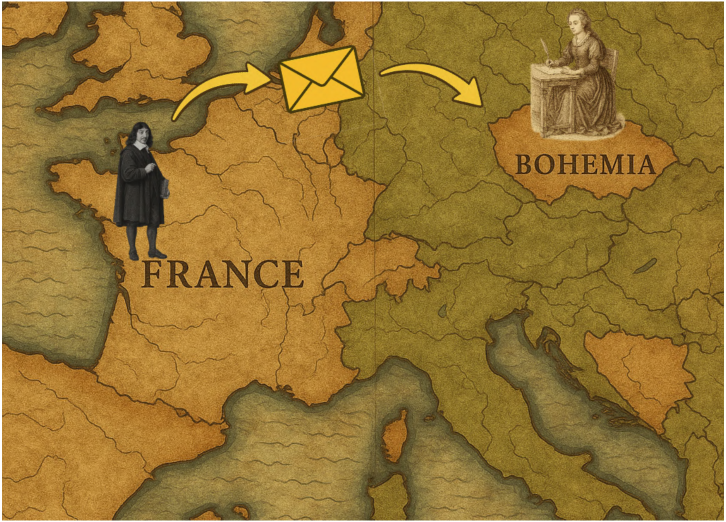




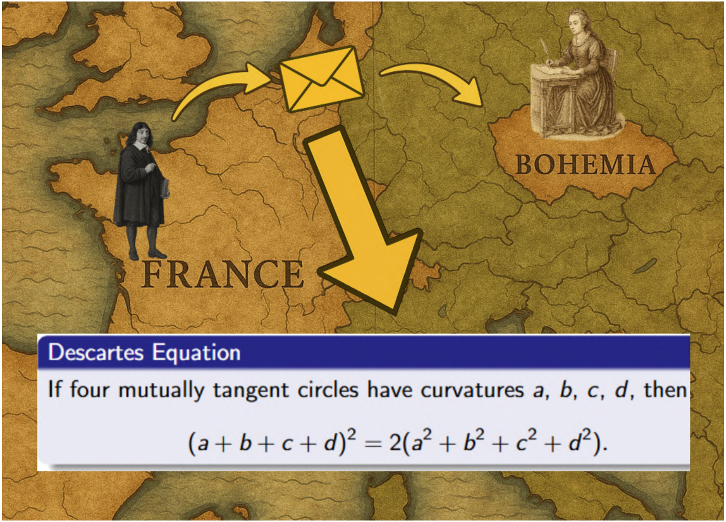
Circle with infinite radius

Descartes' Theorem

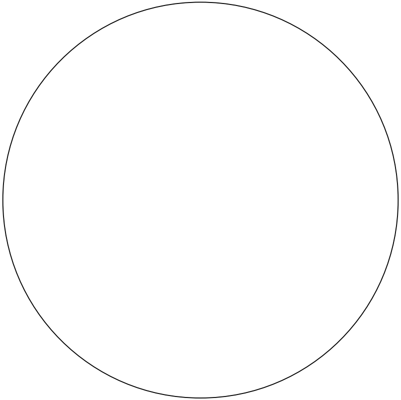
Descartes' Theorem

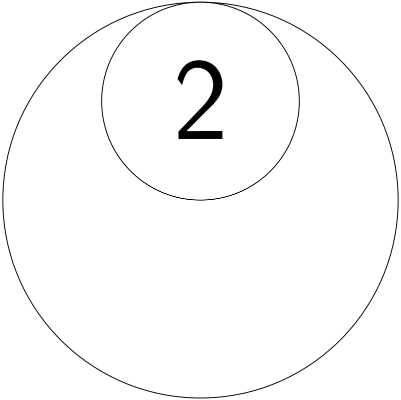


Descartes' Theorem

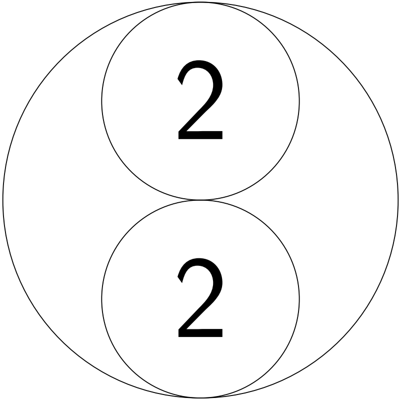


Descartes

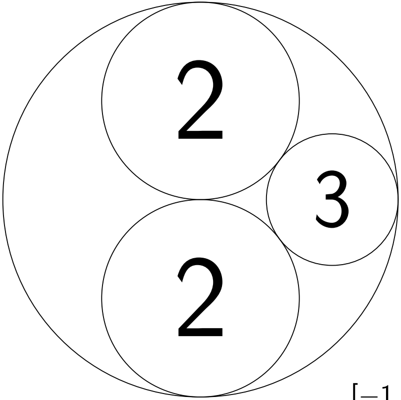



2

Descartes

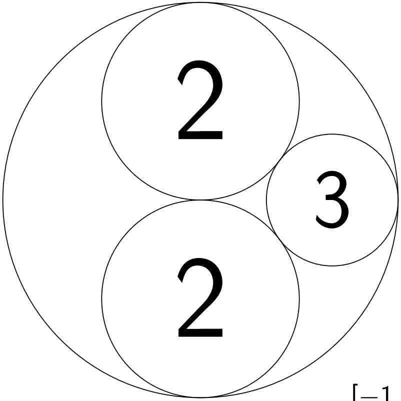


Descartes

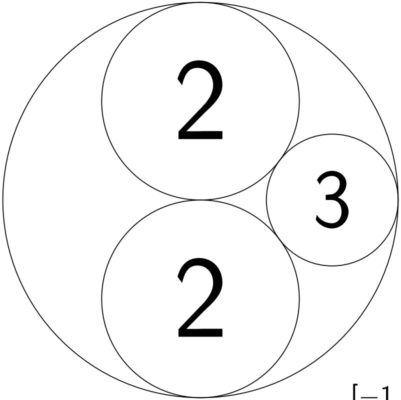


$[-1, 2, 2, 3]$

Descartes



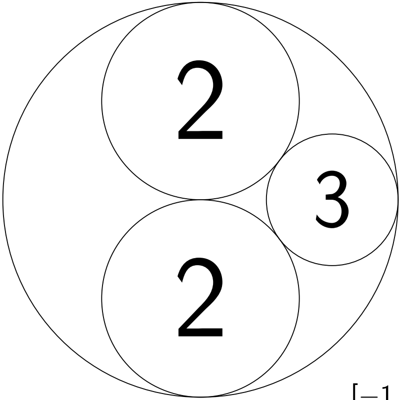
$[-1, 2, 2, 3]$



$[-1, 2, 2, 3]$

$$(-1+2+2+3)^2 \quad = \quad 2(-1^2+2^2+2^2+3^2)$$

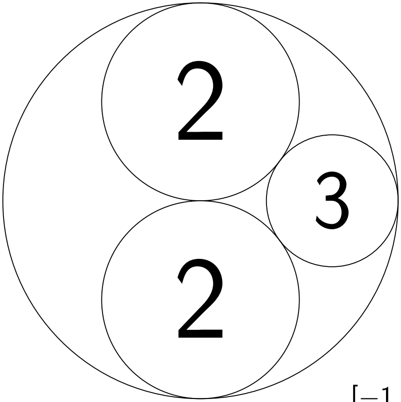
Descartes



$[-1, 2, 2, 3]$

$$
\begin{aligned}
(-1+2+2+3)^2 &= 2(-1^2+2^2+2^2+3^2) \\
6^2 &= 2(1+4+4+9)
\end{aligned}
$$

Descartes

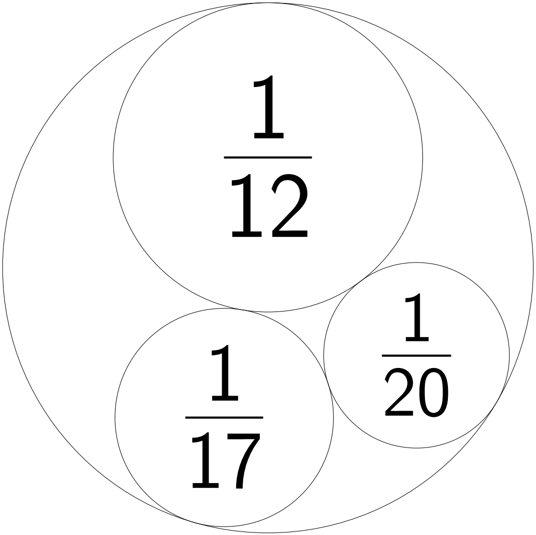


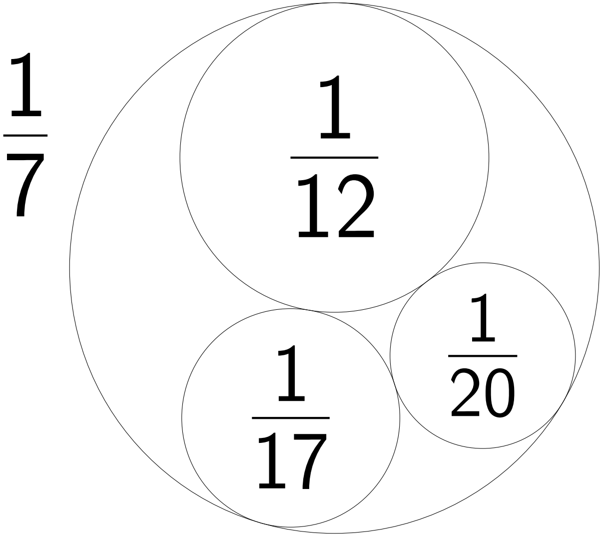
$[-1, 2, 2, 3]$

$$
\begin{aligned}
(-1+2+2+3)^2 &= 2(-1^2+2^2+2^2+3^2) \\
6^2 &= 2(1+4+4+9) \\
36 &= 2 * 18
\end{aligned}
$$

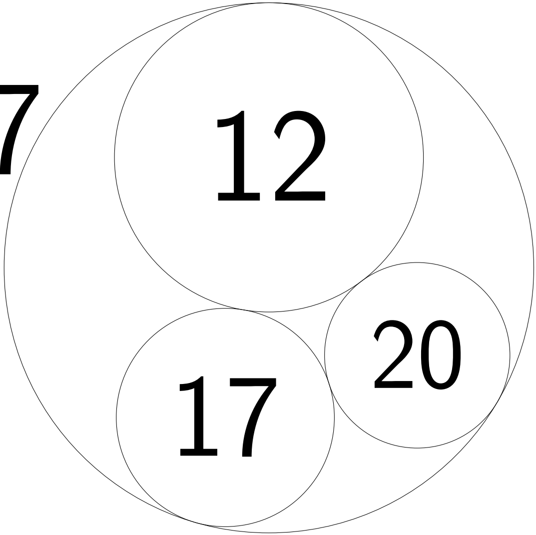
Circle Packing

Circle Packing

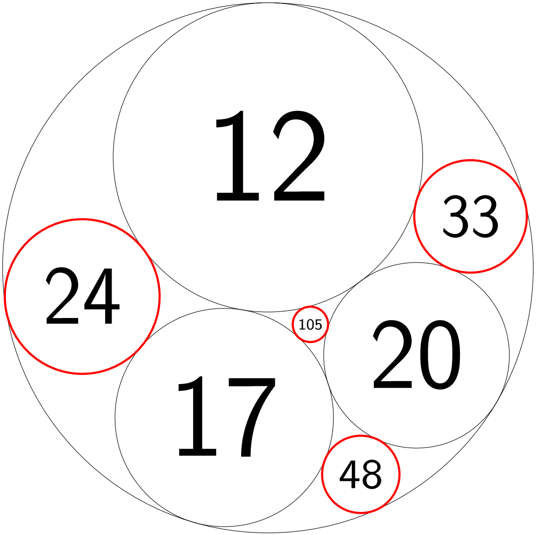



1/7
1/12
1/17
1/20

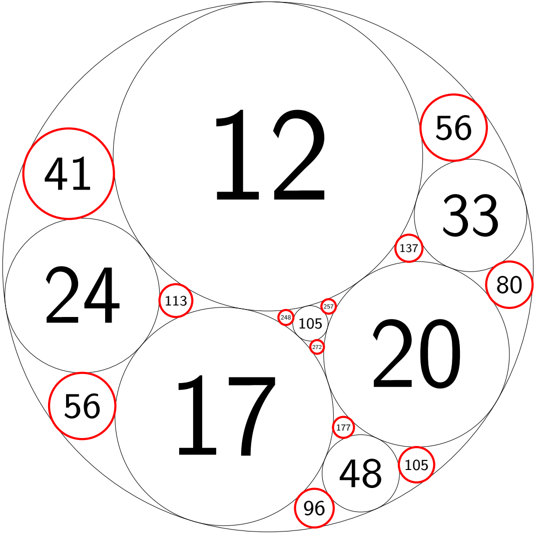
-7



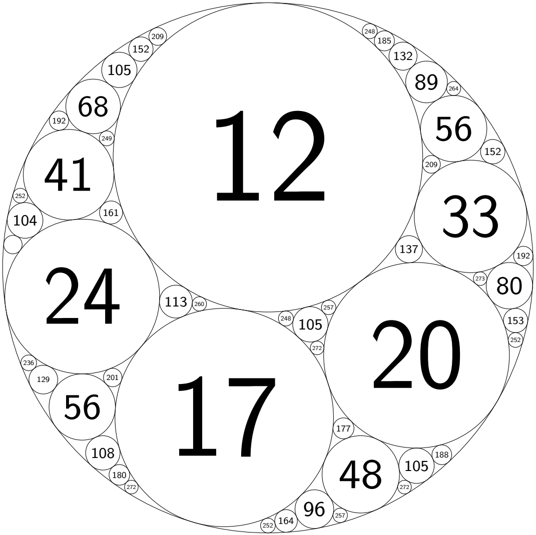
Circle Packing



Circle Packing

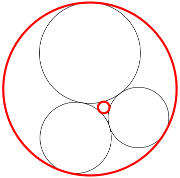


Circle Packing

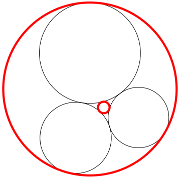


The Descartes Equation

The Descartes Equation

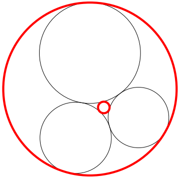


The Descartes Equation



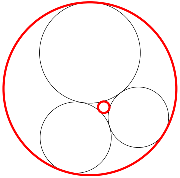
$$2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2 \quad = \quad 0$$

The Descartes Equation



$$\begin{aligned} 2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2 &= 0 \\ d^2-2d(a+b+c)+\left(a^2+b^2+c^2-2ab-2bc-2ac\right) &= 0 \end{aligned}$$

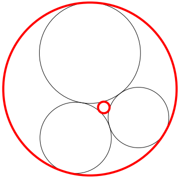
The Descartes Equation



$$
\begin{aligned}
2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2 &= 0\\
d^2-2d(a+b+c)+\left(a^2+b^2+c^2-2ab-2bc-2ac\right) &= 0
\end{aligned}
$$

The quadratic formula gives

The Descartes Equation



$$
\begin{aligned}
2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2 &= 0\\
d^2-2d(a+b+c)+\left(a^2+b^2+c^2-2ab-2bc-2ac\right) &= 0
\end{aligned}
$$

The quadratic formula gives

$$
\begin{aligned}
d &= (a+b+c)\\
&\quad \pm \frac{\sqrt{4(a+b+c)^2-4\left(a^2+b^2+c^2-2ab-2bc-2ac\right)}}{2}\\
&= a+b+c\pm 2\sqrt{ab+bc+ca}.
\end{aligned}
$$

The Descartes Equation

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$

$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Moreover, $d + d' = 2(a + b + c)$.

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Moreover, $d + d' = 2(a + b + c)$.

The Key Relation

$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

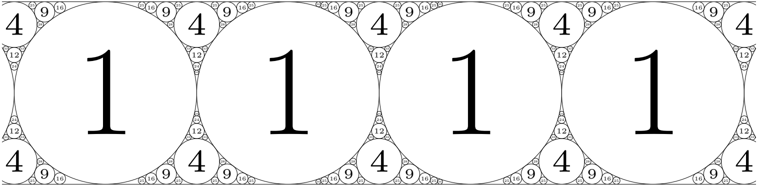
$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Moreover, $d + d' = 2(a + b + c)$.

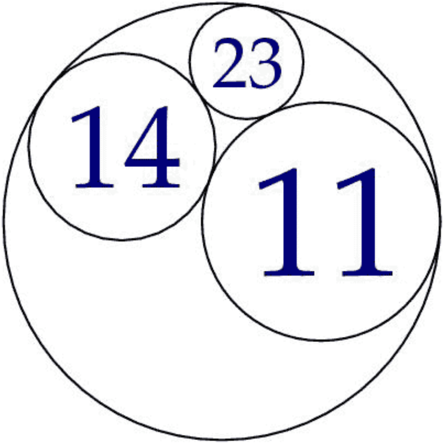
The Key Relation

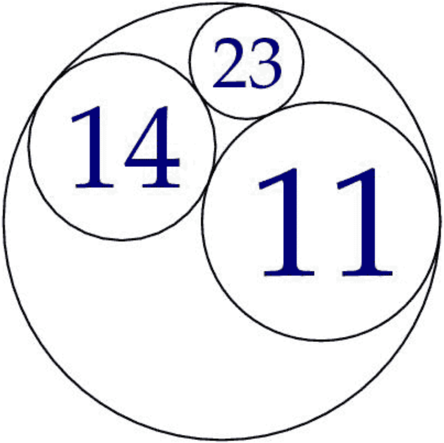
$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$

If a, b, c, d are integers, the rest are also integers!

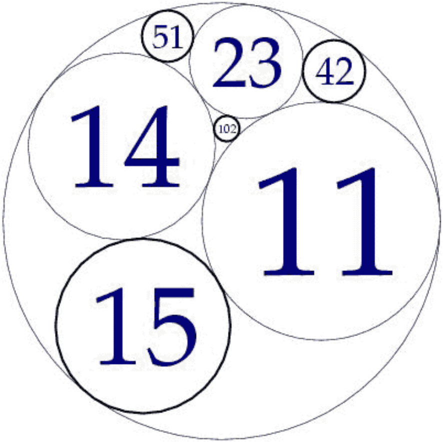


The strip packing: $[0, 0, 1, 1]$

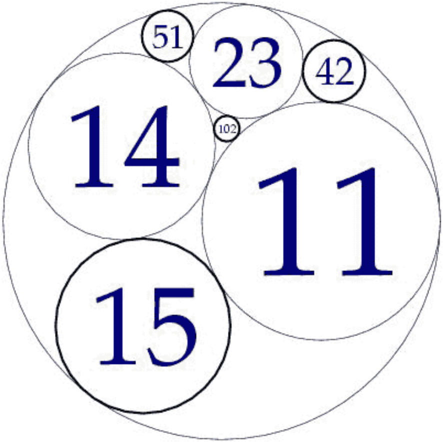

23
14
11



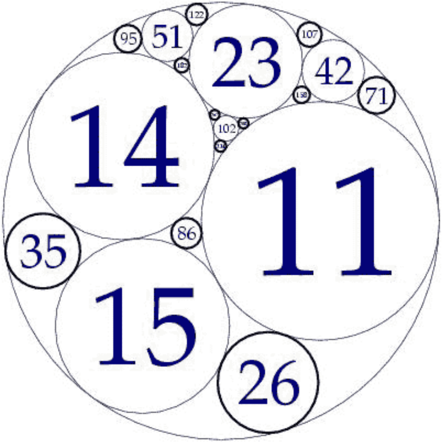
$[-6, 11, 14, 23]$



$[-6, 11, 14, 23]$

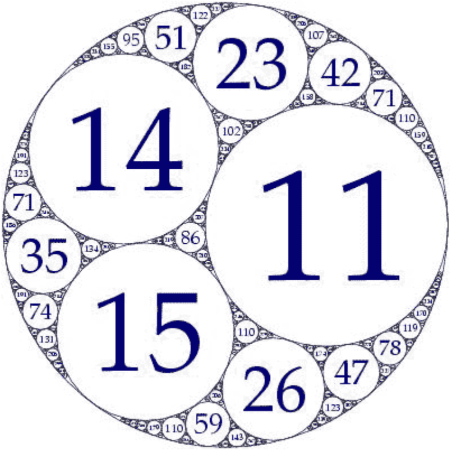


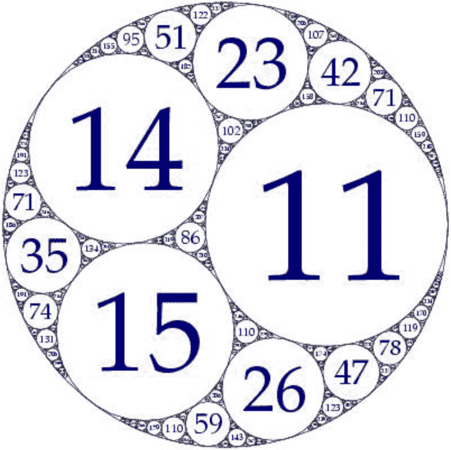
$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$



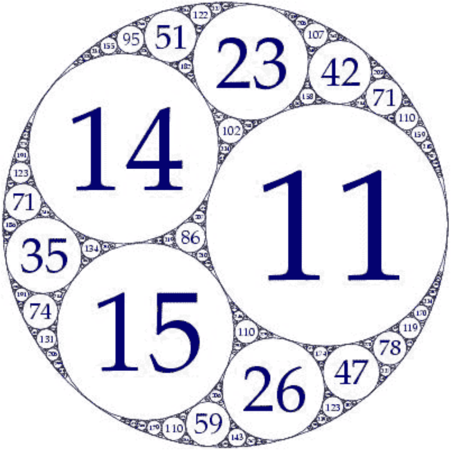
$[-6, 11, 14, 15]$

Apollonian Circle Packings





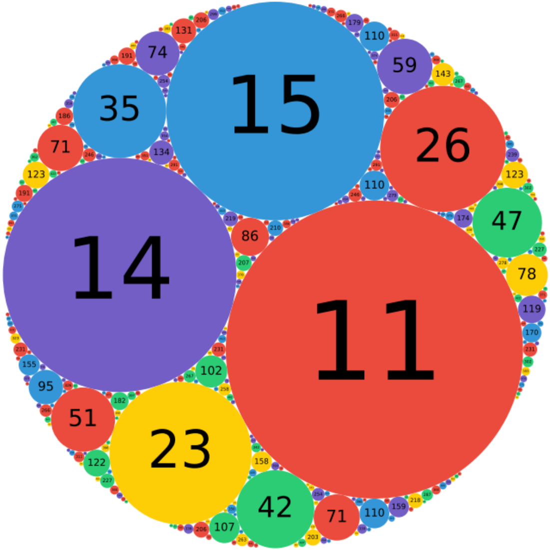
Once $-6, 11, 14, 15$ are set



Once $-6, 11, 14, 15$ are set, no room for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, . . .

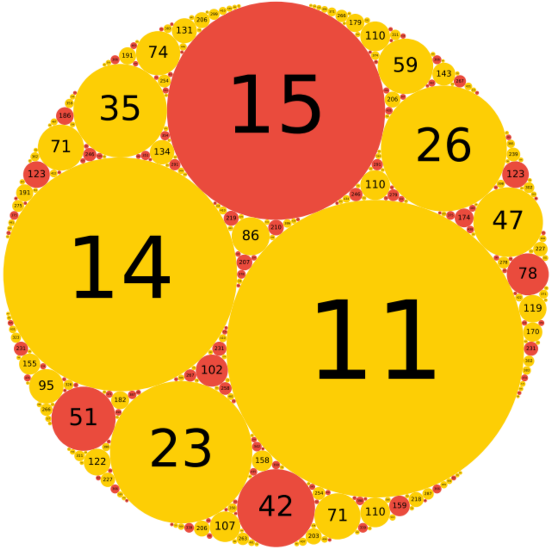
Curvatures Mod 5

Curvatures Mod 5



Curvatures Mod 3

Curvatures Mod 3



Allowed Residues

Allowed Residues

Theorem (Fuchs)

If a congruence obstruction appears, then it appears modulo 24.

Allowed Residues

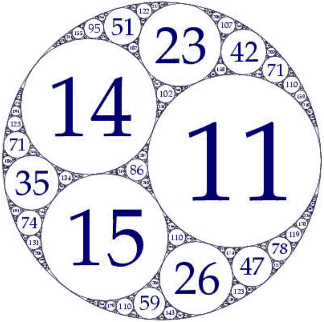
Theorem (Fuchs)

If a congruence obstruction appears, then it appears modulo 24.

Type	Allowed Residues
$(6, 1)$	$0, 1, 4, 9, 12, 16$
$(6, 5)$	$0, 5, 8, 12, 20, 21$
$(6, 13)$	$0, 4, 12, 13, 16, 21$
$(6, 17)$	$0, 8, 9, 12, 17, 20$
$(8, 7)$	$3, 6, 7, 10, 15, 18, 19, 22$
$(8, 11)$	$2, 3, 6, 11, 14, 15, 18, 23$

Allowed Residues

Allowed Residues



$[-6, 11, 14, 15]$

Type	Allowed Residues
$(6, 1)$	$0, 1, 4, 9, 12, 16$
$(6, 5)$	$0, 5, 8, 12, 20, 21$
$(6, 13)$	$0, 4, 12, 13, 16, 21$
$(6, 17)$	$0, 8, 9, 12, 17, 20$
$(8, 7)$	$3, 6, 7, 10, 15, 18, 19, 22$
$(8, 11)$	$2, 3, 6, 11, 14, 15, 18, 23$

Missing Curvatures?

Missing Curvatures?

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } [-1, 2, 2, 3]$$
$$5 \cdot 10^8 \text{ for } [-11, 21, 24, 28]$$

Missing Curvatures?

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } [-1, 2, 2, 3]$$
$$5 \cdot 10^8 \text{ for } [-11, 21, 24, 28]$$

and observed for $[-11, 21, 24, 28]$, there were still a small number (up to 0.013%) of missing curvatures in the range $\left(4 \cdot 10^8, 5 \cdot 10^8\right)$ for residue classes $0, 4, 12, 16 \bmod 24$.

Local-to-global

Local-to-global

Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden, 2011)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

Local-to-global

Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden, 2011)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

Theorem (Bourgain-Kontorovich, 2014)

The number of missing curvatures up to N is at most $O(N^{1-\varepsilon})$ for some computable $\varepsilon > 0$.

Summer 2023 REU

Summer 2023 REU

Summer 2023 REU

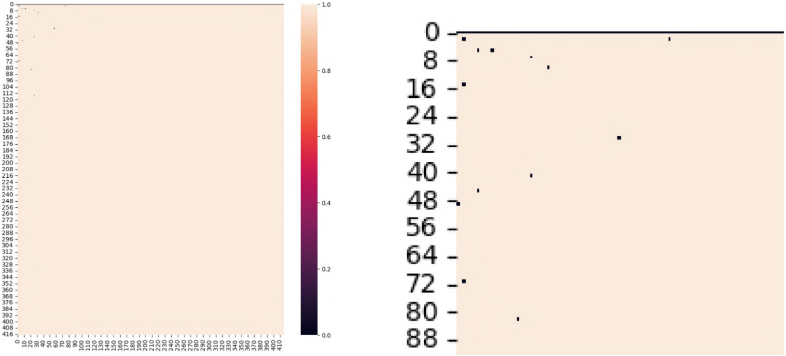
1. Fix a pair of curvatures, and study what packings contain them.

1. Fix a pair of curvatures, and study what packings contain them.
2. Plot: for an admissible pair of residue classes modulo 24, black dot if no packing has that pair.

1. Fix a pair of curvatures, and study what packings contain them.
2. Plot: for an admissible pair of residue classes modulo 24, black dot if no packing has that pair.
3. Local-global: finitely many black dots on any row or column.

Typical graph

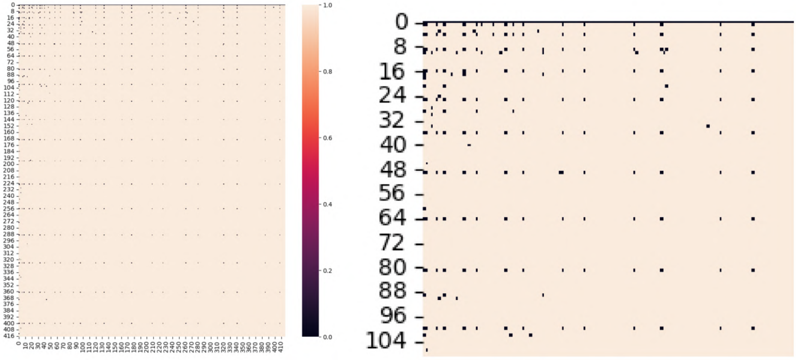
Typical graph



Residue classes 0 (mod 24) and 12 (mod 24) (Summer Haag)

One weird graph

One weird graph



Residue classes 0 (mod 24) and 8 (mod 24) (Summer Haag)

Where's the bug?

Where's the bug?

Where's the bug?

No bug–The conjecture is false!

No bug–The conjecture is false!

Theorem (Haag-Kertzer-Rickards-Stange)

The packing $[-3, 5, 8, 8]$ *has no square curvatures.*

How did we prove it? Quadratic Forms

How did we prove it? Quadratic Forms

There is a bijection between

How did we prove it? Quadratic Forms

There is a bijection between

1. curvatures of circles tangent to fixed outer circle of curvature, and

How did we prove it? Quadratic Forms

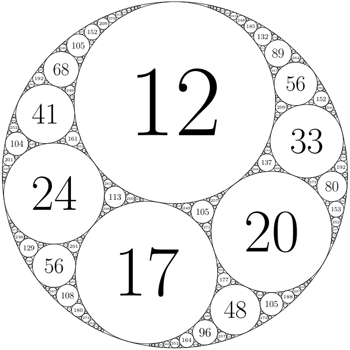
There is a bijection between

1. curvatures of circles tangent to fixed outer circle of curvature, and
2. $\{f_a(x, y) - a : \gcd(x, y) = 1\}$

How did we prove it? Quadratic Forms

There is a bijection between

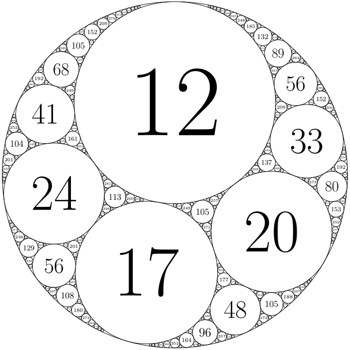
1. curvatures of circles tangent to fixed outer circle of curvature, and
2. $\{f_a(x,y) - a : \gcd(x,y) = 1\}$
3. ex. in $[-7, 12, 17, 20]$, fix 17 and 20:



How did we prove it? Quadratic Forms

There is a bijection between

1. curvatures of circles tangent to fixed outer circle of curvature, and
2. $\{f_a(x,y) - a : \gcd(x,y) = 1\}$
3. ex. in $[-7, 12, 17, 20]$, fix 17 and 20:



x	$f(x) = 29x^2 - 56x + 20$
-1	105
0	20
1	-7
2	24
3	113

New idea

New idea

1. All curvatures n in $[-3, 5, 8, 8]$ have $n \equiv 0, 1 \pmod 4$.

New idea

1. All curvatures n in $[-3, 5, 8, 8]$ have $n \equiv 0, 1 \pmod 4$.
2. Fix circle of curvature n; tangent curvatures $f(x, y) - n$

New idea

1. All curvatures n in $[-3, 5, 8, 8]$ have $n \equiv 0, 1 \pmod 4$.
2. Fix circle of curvature n; tangent curvatures $f(x, y) - n$
3. Modulo n and equivalence, values are Ax^2: only quadratic residues or only non-residues.

New idea

1. All curvatures n in $[-3, 5, 8, 8]$ have $n \equiv 0, 1 \pmod 4$.
2. Fix circle of curvature n; tangent curvatures $f(x, y) - n$
3. Modulo n and equivalence, values are Ax^2: only quadratic residues or only non-residues.
4. Define $\chi_2(\mathcal{C}) = 1$ if solution exists, -1 otherwise.

New idea

1. Suppose that $\mathcal{C}_1, \mathcal{C}_2$ in a packing are tangent, having non-zero coprime curvatures a and b respectively.

New idea

1. Suppose that $\mathcal{C}_1, \mathcal{C}_2$ in a packing are tangent, having non-zero coprime curvatures a and b respectively.
2. Then

$$\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = 1 \implies \chi_2(\mathcal{C}_1) = \chi_2(\mathcal{C}_2).$$

New idea

1. Suppose that $\mathcal{C}_1, \mathcal{C}_2$ in a packing are tangent, having non-zero coprime curvatures a and b respectively.
2. Then
$$\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = 1 \implies \chi_2(\mathcal{C}_1) = \chi_2(\mathcal{C}_2).$$
3. Any two circles in the packing are connected by a path of pairwise coprime curvatures.

New idea

1. Suppose that $\mathcal{C}_1, \mathcal{C}_2$ in a packing are tangent, having non-zero coprime curvatures *a* and *b* respectively.
2. Then
$$\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = 1 \implies \chi_2(\mathcal{C}_1) = \chi_2(\mathcal{C}_2).$$
3. Any two circles in the packing are connected by a path of pairwise coprime curvatures.
4. So $\chi_2(\mathcal{C})$ is independent of the choice of circle $\mathcal{C}$.

There are no squares in the packing

There are no squares in the packing

1. In base quadruple $[-3, 5, 8, 8]$, compute

$$\chi_2(\text{a packing}) = \text{is 8 a quadratic residue mod 5?}$$

There are no squares in the packing

1. In base quadruple $[-3, 5, 8, 8]$, compute

$$\chi_2(\text{a packing}) = \text{is 8 a quadratic residue mod 5?} \implies \text{no} = -1.$$

There are no squares in the packing

1. In base quadruple $[-3, 5, 8, 8]$, compute

$$\chi_2(\text{a packing}) = \text{is 8 a quadratic residue mod 5?} \implies \text{no} = -1.$$

2. So no circle can be tangent to a square.

New invariants of a packing

New invariants of a packing

$$\chi_2 : \{\text{circles}\} \to \{\pm 1\}$$

constant across a packing

New invariants of a packing

$$\chi_2 : \{\text{circles}\} \to \{\pm 1\}$$

constant across a packing

$$\chi_4 : \{\text{circles in packing of type } (6,1) \text{ or } (6,17)\} \to \{1, i, -1, -i\}$$

satisfies $\chi_4(\mathcal{C})^2 = \chi_2(\mathcal{C})$,
constant across a packing.

New invariants of a packing

$$\chi_2 : \{\text{circles}\} \to \{\pm 1\}$$

constant across a packing

$$\chi_4 : \{\text{circles in packing of type } (6,1) \text{ or } (6,17)\} \to \{1, i, -1, -i\}$$

satisfies $\chi_4(\mathcal{C})^2 = \chi_2(\mathcal{C})$,
constant across a packing.

The values of χ_2 and χ_4 determine the quadratic and quartic obstructions respectively.

The New Conjecture

The New Conjecture

The type of a packing implies the existence of certain quadratic and quartic obstructions:

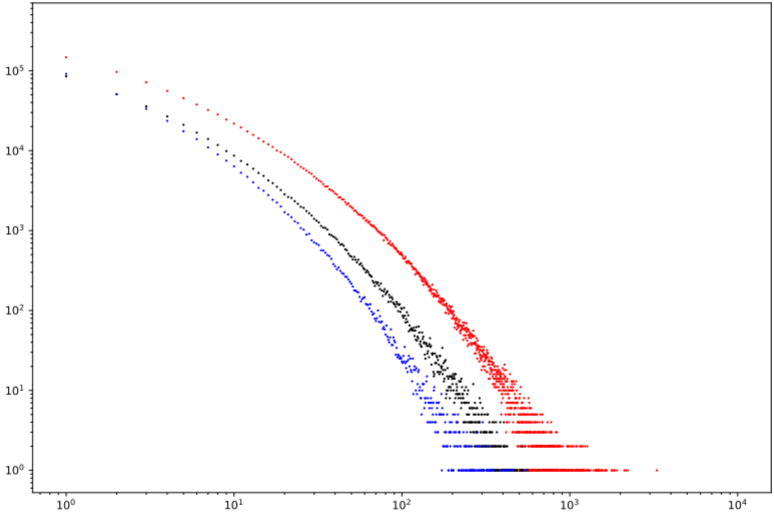
The New Conjecture

The type of a packing implies the existence of certain quadratic and quartic obstructions:

Type	n^2 Obstructions	n^4 Obstructions	L-G false	L-G open
$(6,1,1,-1)$		$n^4, 4n^4, 9n^4, 36n^4$	$0,1,4,9,12,16$	
$(6,1,-1)$	$n^2, 2n^2, 3n^2, 6n^2$		$0,1,4,9,12,16$	
$(6,5,1)$	$2n^2, 3n^2$		$0,8,12$	$5,20,21$
$(6,5,-1)$	$n^2, 6n^2$		$0,12$	$5,8,20,21$
$(6,13,1)$	$2n^2, 6n^2$		0	$4,12,13,16,21$
$(6,13,-1)$	$n^2, 3n^2$		$0,4,12,16$	$13,21$
$(6,17,1,1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0,9,12$	$8,17,20$
$(6,17,1,-1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0,9,12$	$8,17,20$
$(6,17,-1)$	$n^2, 2n^2$		$0,8,9,12$	$17,20$
$(8,7,1)$	$3n^2, 6n^2$		$3,6$	$7,10,15,18,19,22$
$(8,7,-1)$	$2n^2$		18	$3,6,7,10,15,19,22$
$(8,11,-1)$	$2n^2, 3n^2, 6n^2$		$2,3,6,18$	$11,14,15,23$

Sporadic curvatures dropping off

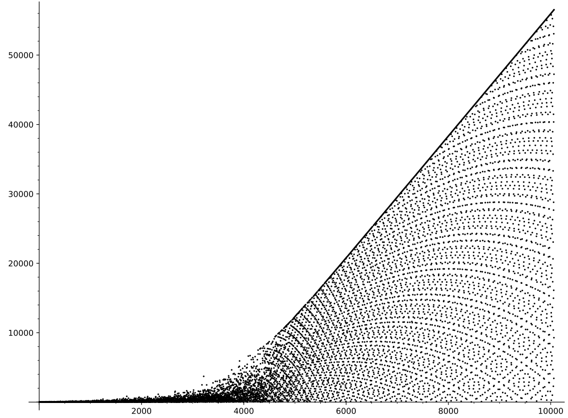
Sporadic curvatures dropping off



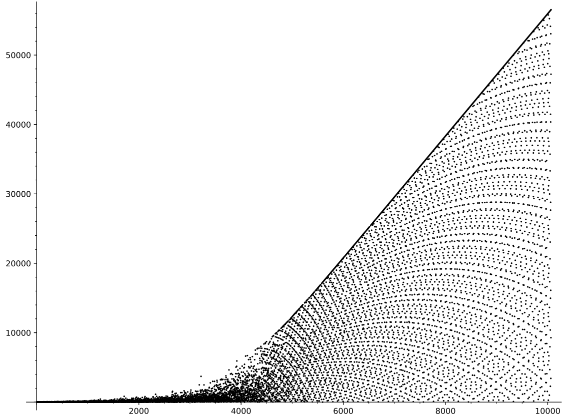
A loglog plot of the probability a curvature is sporadic, as curvature size increases.

Successive differences

Successive differences

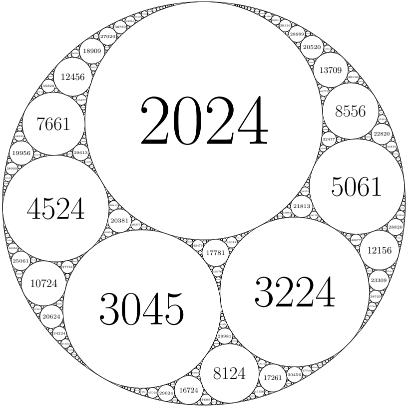


Successive differences



Successive differences of missing curvatures in the packing $[-4, 5, 20, 21]$. The quadratic families $2n^2$ and $3n^2$ begin to predominate (the sporadic set has 3659 elements $< 10^{10}$, and occur increasingly sparsely.)

Thank You!


2024
3045
3224
4524
5061
7661
8124
8556
10724
12156
12456
13709
16724
17261
17781
18909
19956
20381
20520
20624
21813
22820
23309
25061
27020
28989
29024
29613
29981
30456