Packing Problems & Number Theory

Clyde Kertzer

University of Colorado Boulder

May 7, 2025

A classic game

A classic game

A guessing jar:

A classic game

A guessing jar: How many marbles?

A classic game

A guessing jar: How many marbles? 223!

A classic game

A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?

A classic game

A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?
2. Given 223 marbles, what's the smallest container holding them all?

A classic game

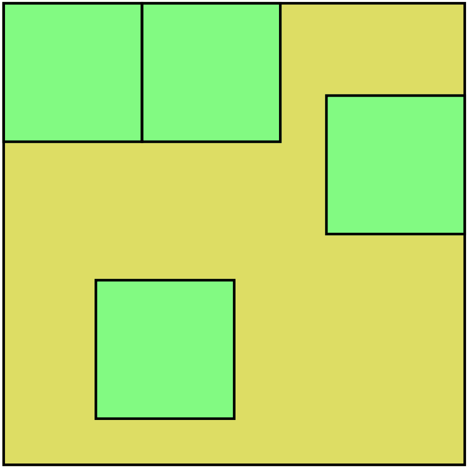
A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?
2. Given 223 marbles, what's the smallest container holding them all?

We need to simplify the problem...

Square Packing

Square Packing



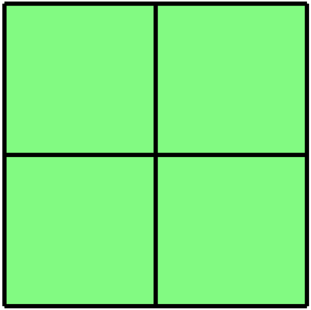
Square Packing

Square Packing

What's the smallest square we can fit 4 squares inside of?

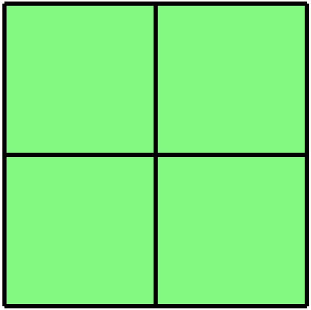
Square Packing

What's the smallest square we can fit 4 squares inside of?



Square Packing

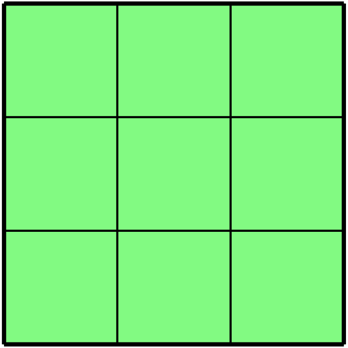
What's the smallest square we can fit 4 squares inside of?



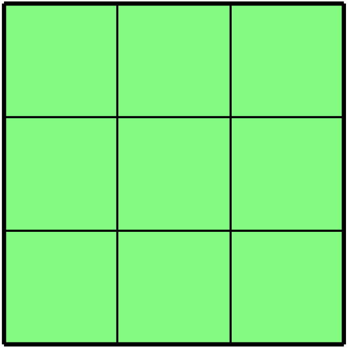
Side length: 2

What's the smallest square we can fit 9 squares inside of?

What's the smallest square we can fit 9 squares inside of?



What's the smallest square we can fit 9 squares inside of?



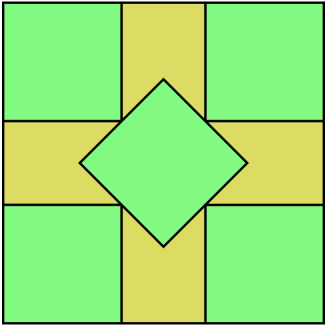
Side length: 3

Square Packing

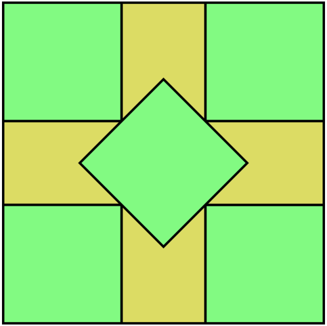
Square Packing

What about 5 squares?

What about 5 squares?



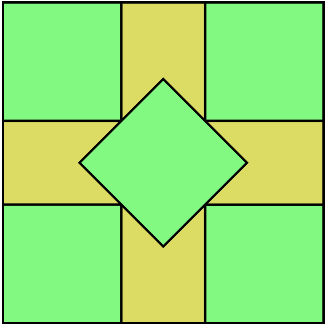
What about 5 squares?



Side length:

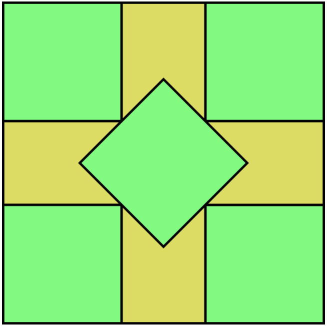
Square Packing

What about 5 squares?



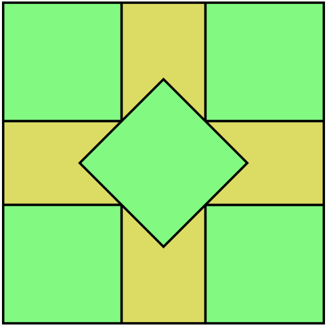
Side length: $\approx 2.707\ldots$

What about 5 squares?



Side length: $\approx 2.707\ldots = 2 + \dfrac{\sqrt{2}}{2}$

What about 5 squares?

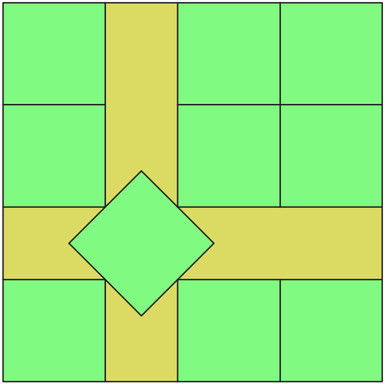


Side length: $\approx 2.707\ldots = 2 + \dfrac{\sqrt{2}}{2}$

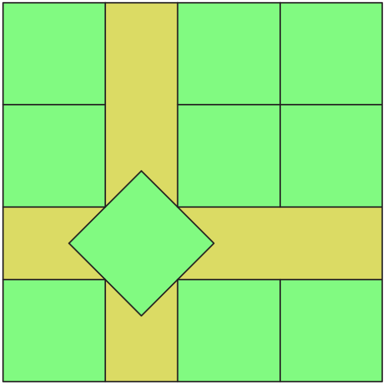
Can we use this packing to find the optimal packing of 10 squares?

Square Packing

Square Packing



Square Packing



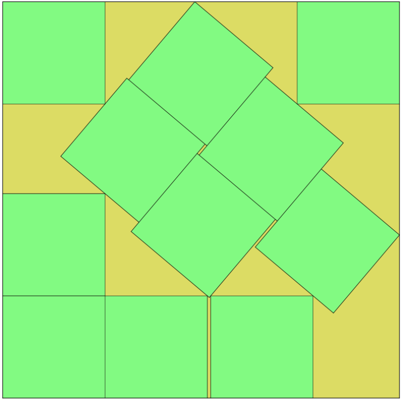
Side length: $\approx 3.707\ldots = 3 + \dfrac{\sqrt{2}}{2}$

Square Packing

Square Packing

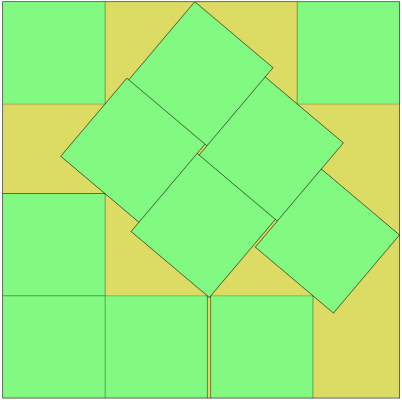
What about 11 squares?

What about 11 squares?



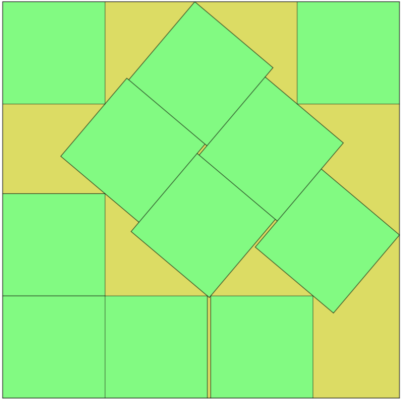
Square Packing

What about 11 squares?



Side length: $\approx 3.877\ldots$

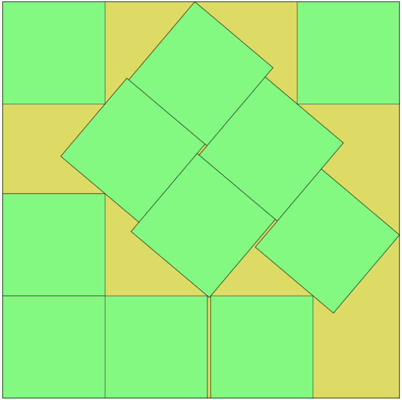
What about 11 squares?



Side length: $\approx 3.877\ldots$

Is this the best possible packing?

What about 11 squares?

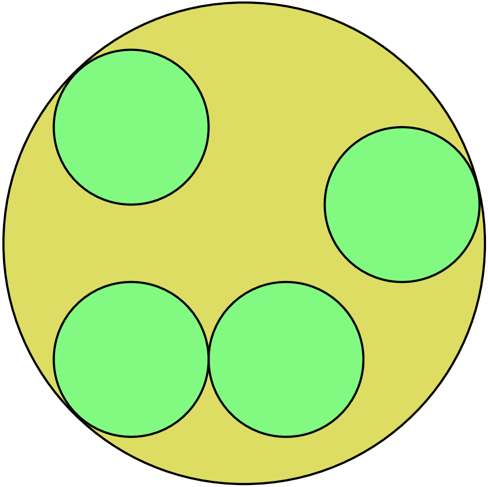


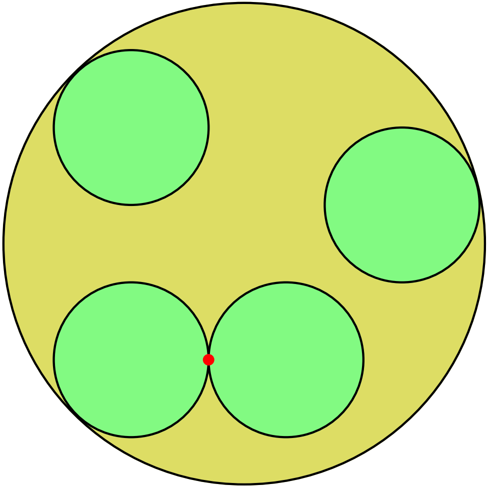
Side length: $\approx 3.877\ldots$

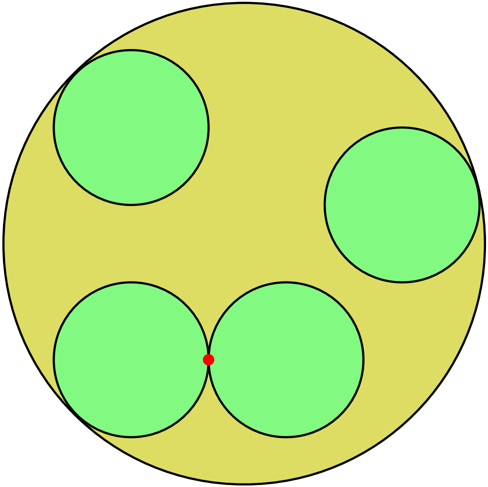
Is this the best possible packing? Mathematicians still don't know...

Circle Packing

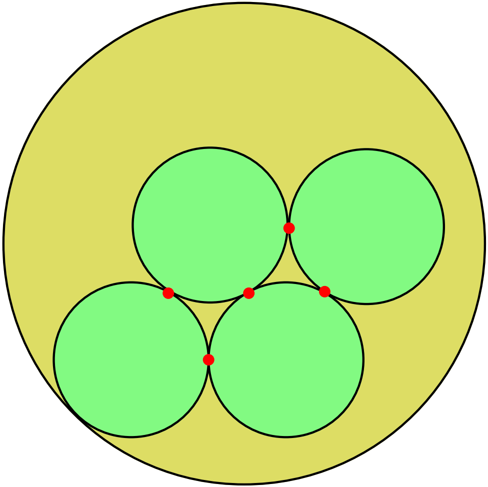
Circle Packing

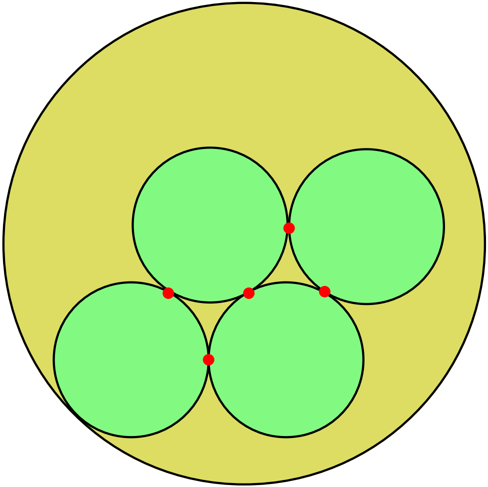






Try it yourself: Can you find an arrangement of 4 circles such that each one is tangent to the other?

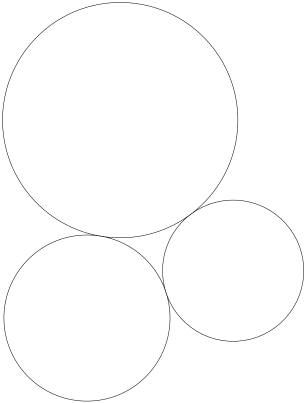




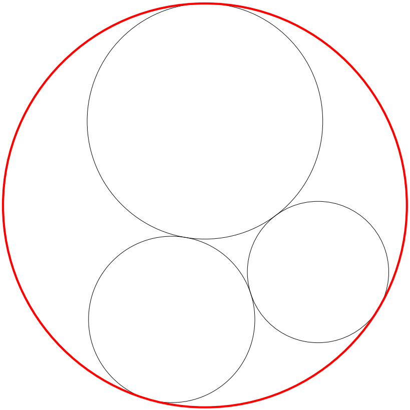
What if the circles aren't all the same size...

Circle Packing

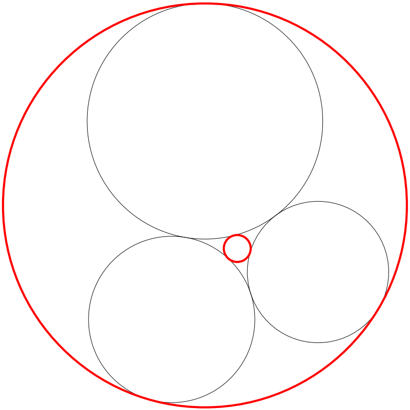
Circle Packing

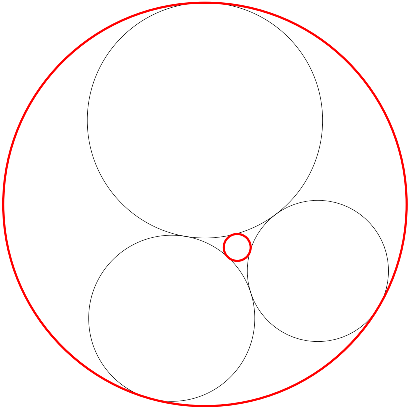


Circle Packing



Circle Packing





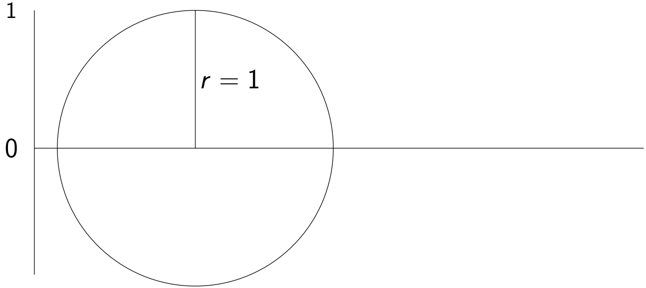
Definition

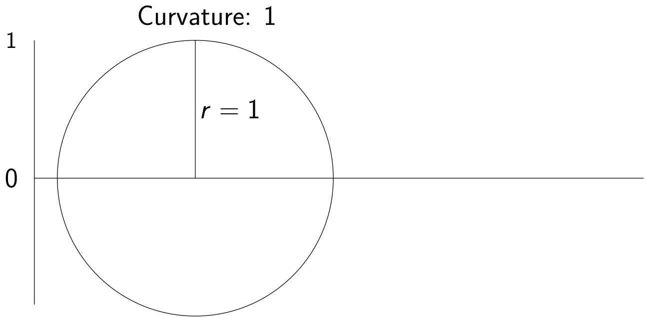
A set of four mutually tangent circles is called a *Descartes Quadruple*

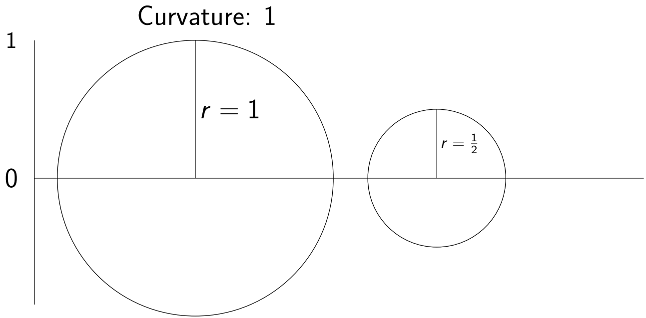
Circle Packing: Curvature

Circle Packing: Curvature

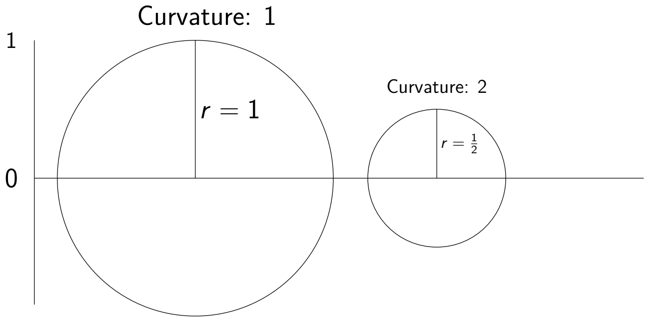
Circle Packing: Curvature



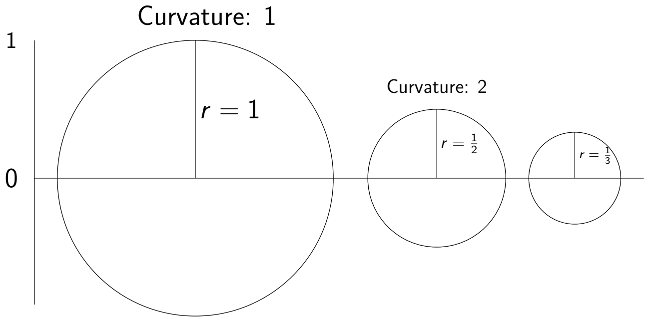

Curvature: 1
1
0
$r = 1$


Curvature: 1
1
0
$r = 1$
$r = \frac{1}{2}$

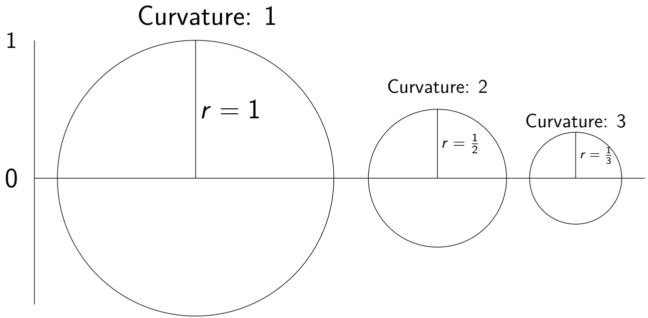
Circle Packing: Curvature



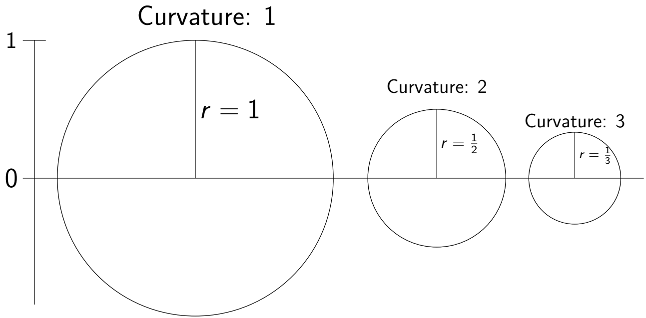
Circle Packing: Curvature



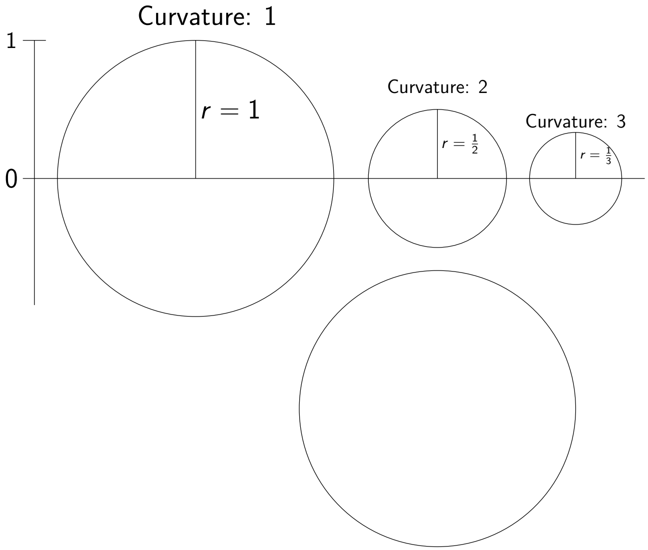
Circle Packing: Curvature



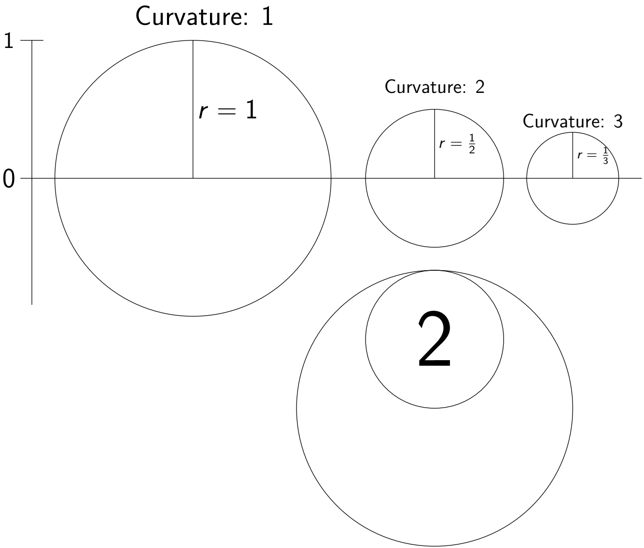
Circle Packing: Curvature



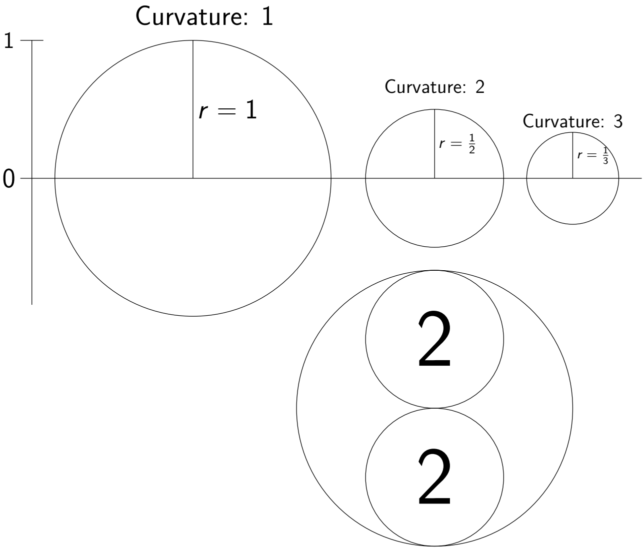
Circle Packing: Curvature



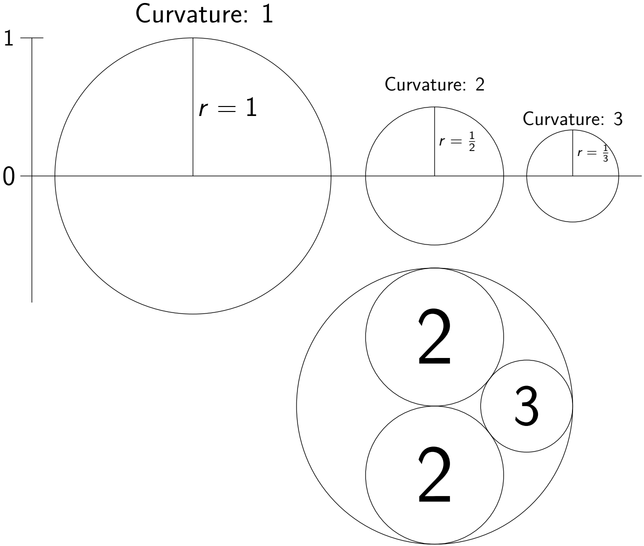
Circle Packing: Curvature



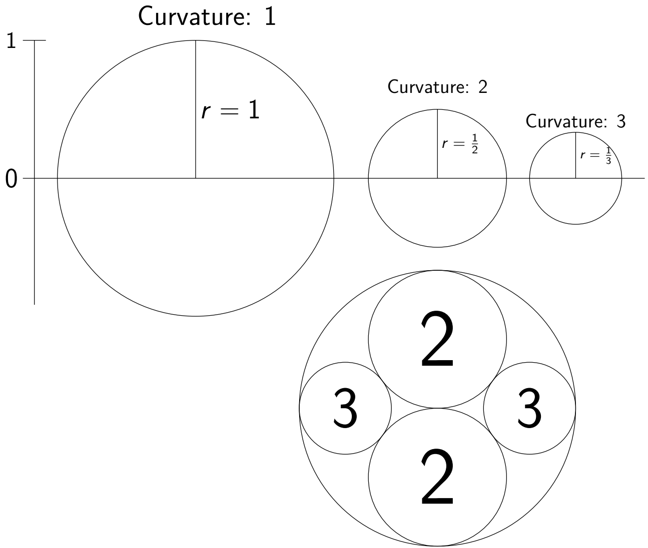
Circle Packing: Curvature



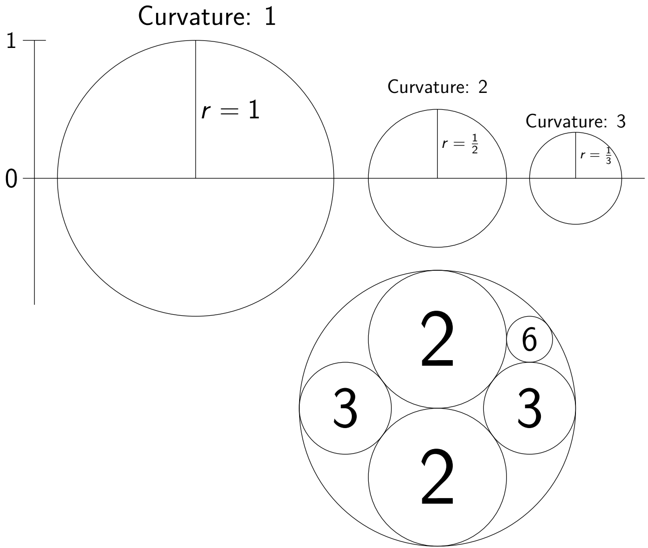
Circle Packing: Curvature



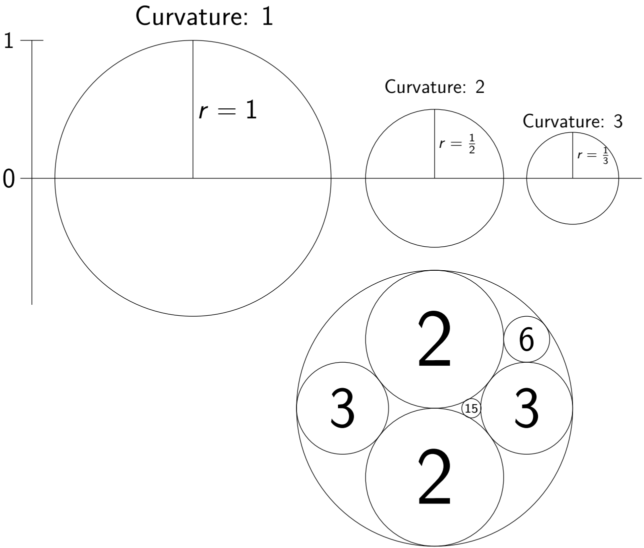
Circle Packing: Curvature

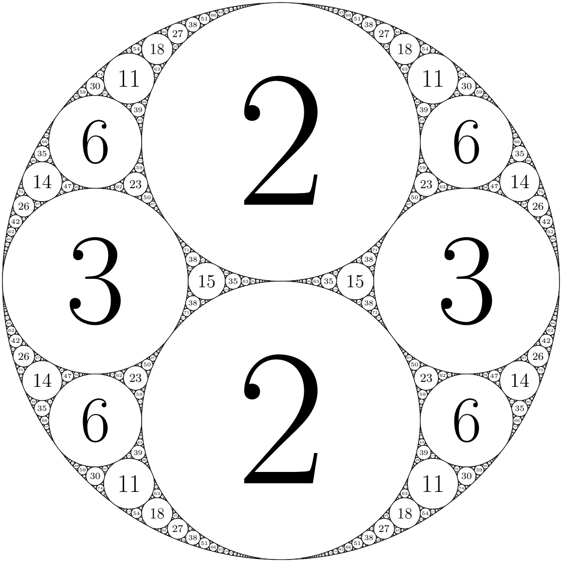


Circle Packing: Curvature

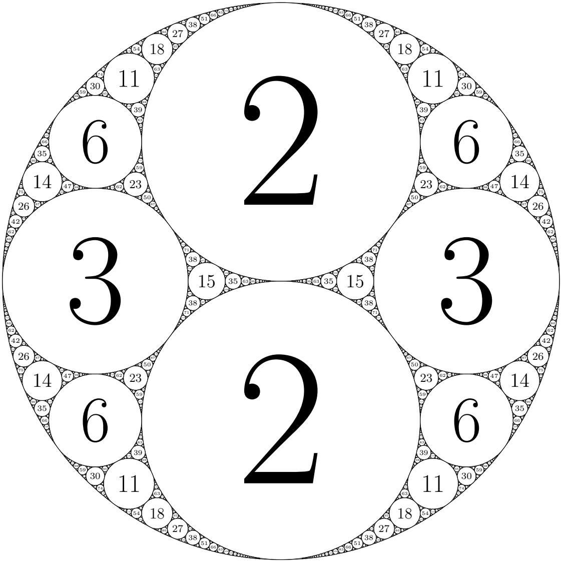


Circle Packing: Curvature



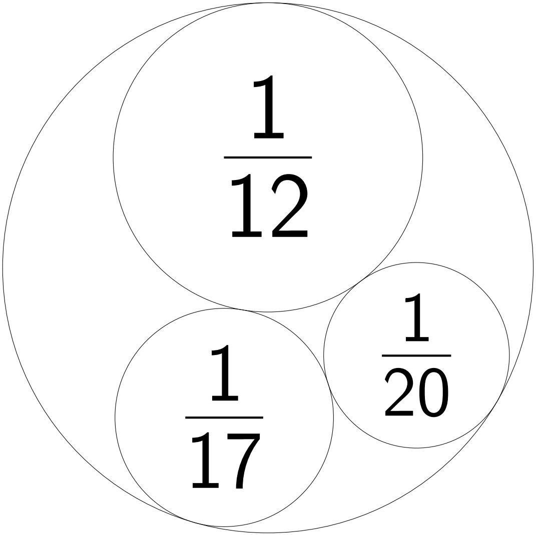

2
2
3
3
6
6
6
6
11
11
11
11
14
14
14
14
15
15
18
18
18
18
23
23
23
23
26
26
26
26
27
27
27
27
30
30
30
30
35
35
35
35
38
38
38
38
39
39
39
39

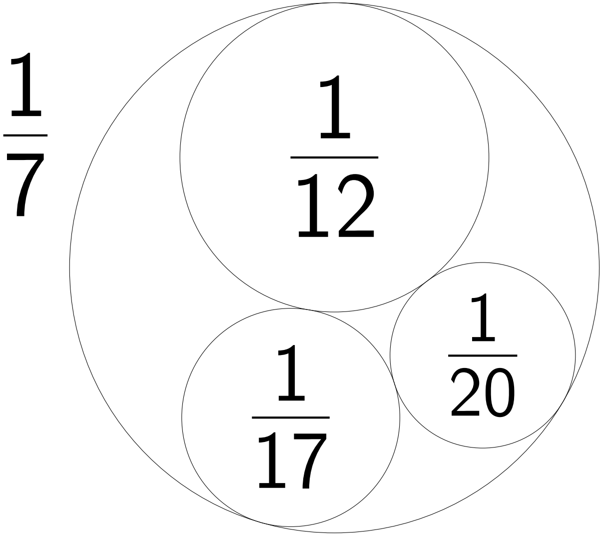
Circle Packing: Curvature



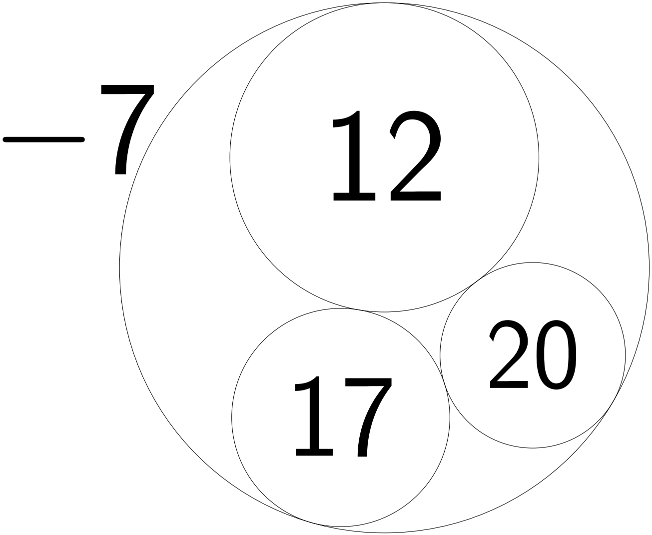
With your set can you make a circle packing that is not symmetric?

Circle Packing

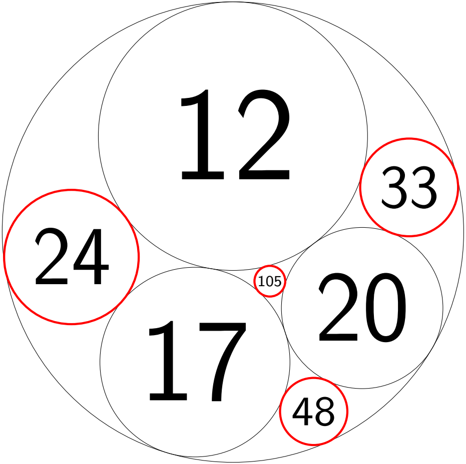



$\frac{1}{7}$
$\frac{1}{12}$
$\frac{1}{17}$
$\frac{1}{20}$

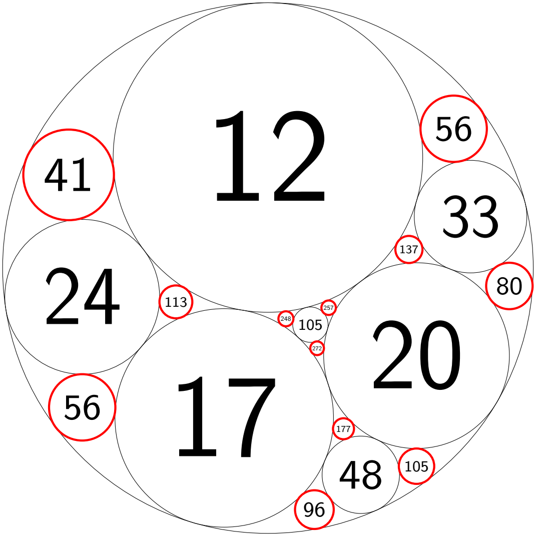
Circle Packing


−7
12
20
17

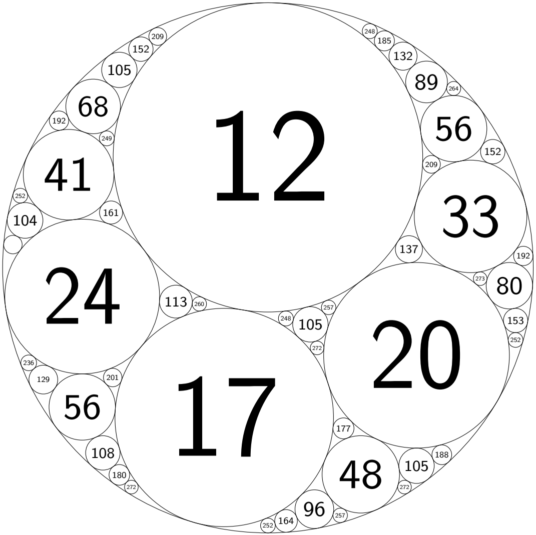
Circle Packing



Circle Packing



Circle Packing



Summary: The Descartes Equation

Summary: The Descartes Equation

Definition

Summary: The Descartes Equation

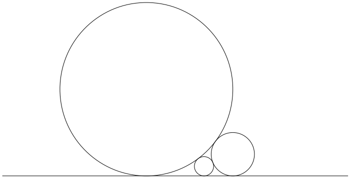
Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

Summary: The Descartes Equation

Definition

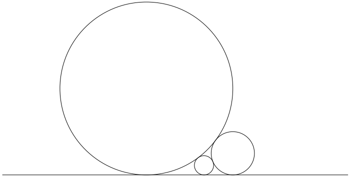
The *curvature* of a circle with radius r is defined to be $1/r$.



Summary: The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

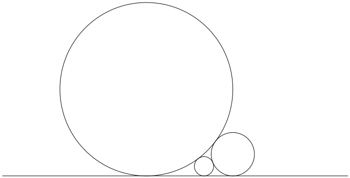


Circle with infinite radius (Curvature 0)

Summary: The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



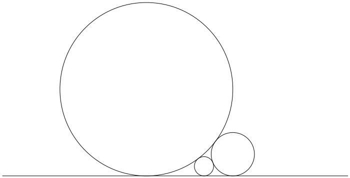
Circle with infinite radius (Curvature 0)

Definition

Summary: The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius (Curvature 0)

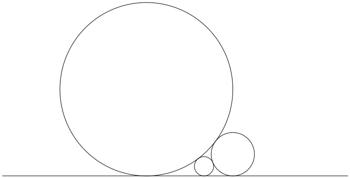
Definition

Four mutually tangent circles are called a *Descartes Quadruple*.

Summary: The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius (Curvature 0)

Definition

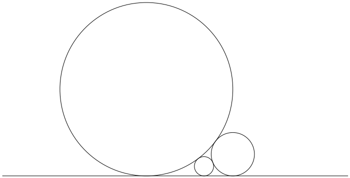
Four mutually tangent circles are called a *Descartes Quadruple*.

Descartes Equation

Summary: The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius (Curvature 0)

Definition

Four mutually tangent circles are called a *Descartes Quadruple*.

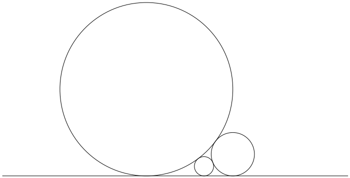
Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

Summary: The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius (Curvature 0)

Definition

Four mutually tangent circles are called a *Descartes Quadruple*.

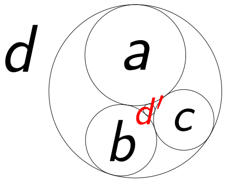
Descartes Equation

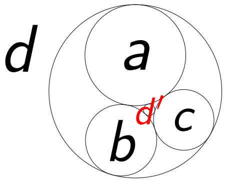
If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

Summary: The Descartes Equation

Summary: The Descartes Equation

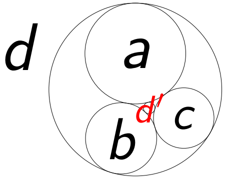




Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

Summary: The Descartes Equation

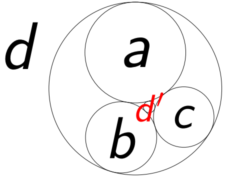


Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d′ have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Summary: The Descartes Equation



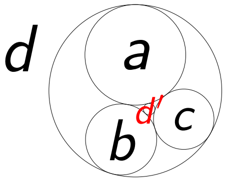
Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d′ have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Key Relation

Summary: The Descartes Equation



Corollary

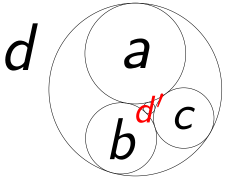
If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Key Relation

$d + d' = 2(a + b + c)$

Summary: The Descartes Equation



Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

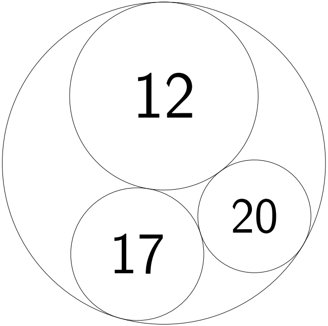
$$d = a + b + c + 2\sqrt{ab + ac + bc}$$

$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

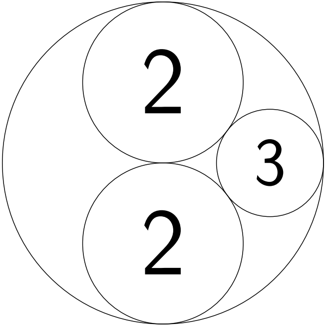
The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

The Descartes Equation

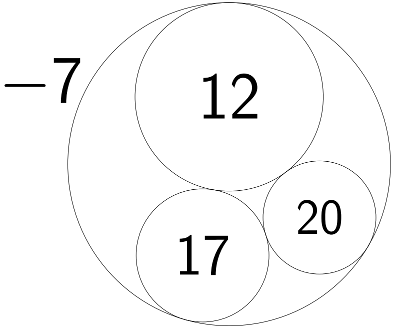


$[-7, 12, 17, 20]$

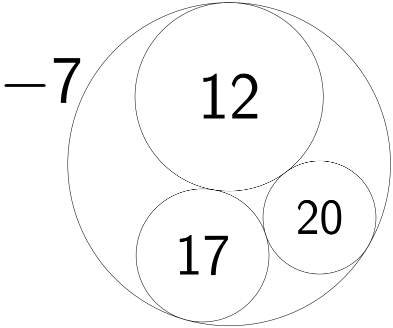


$[-1, 2, 2, 3]$

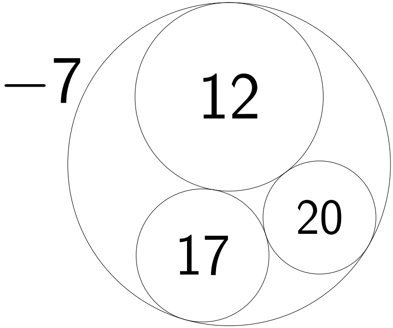
The Descartes Equation



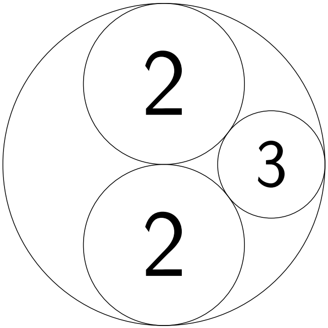
The Descartes Equation



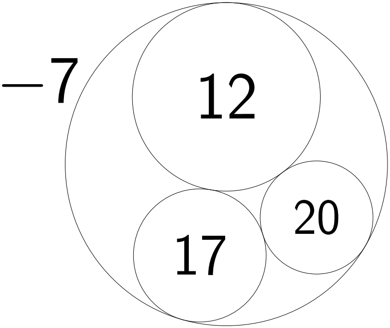
$[-7, 12, 17, 20]$



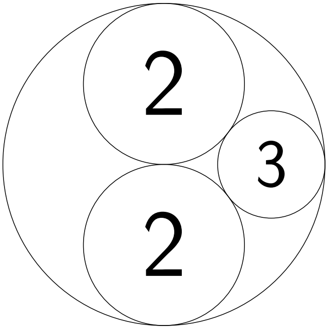
$[-7, 12, 17, 20]$



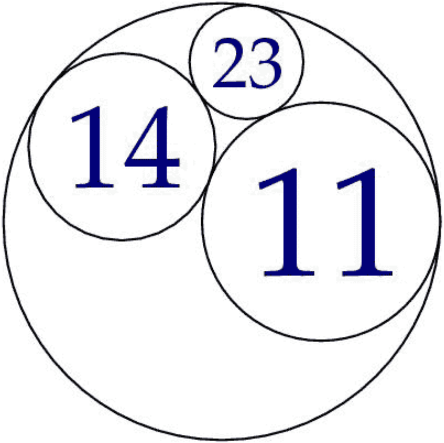
The Descartes Equation

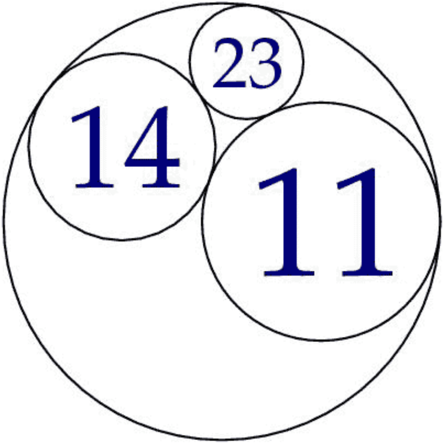


$[-7, 12, 17, 20]$

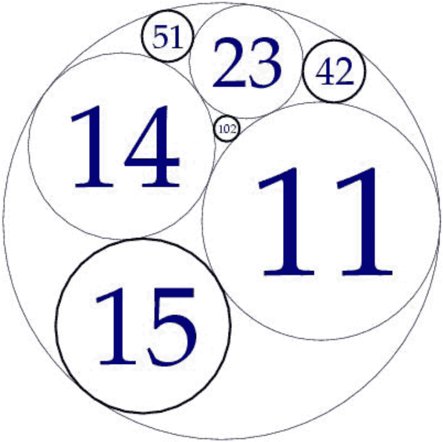


$[-1, 2, 2, 3]$

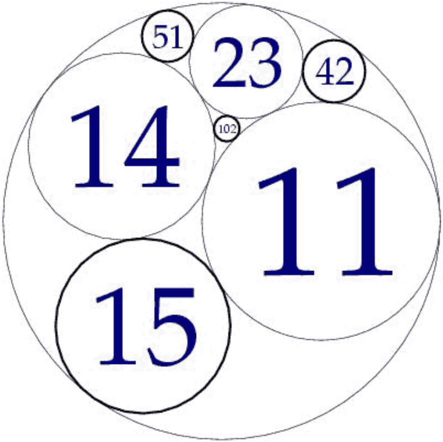

23
14
11



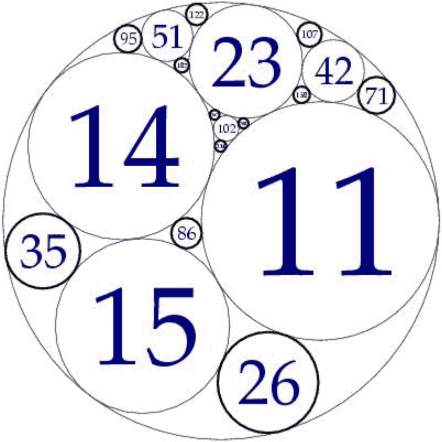
$[-6, 11, 14, 23]$



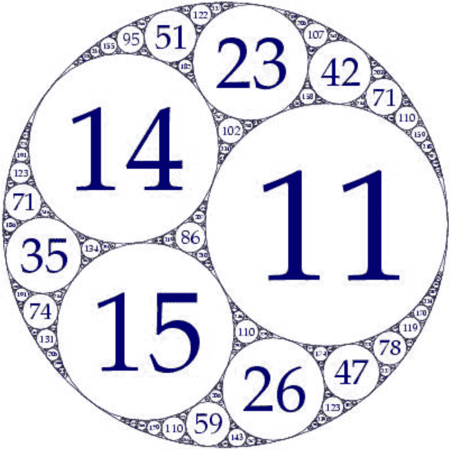
$[-6, 11, 14, 23]$



$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$



$[-6, 11, 14, 15]$


14
23
11
15
26
51
42
95
71
102
86
35
71
123
74
110
47
78
59
110
107
122
110
119
134
131
123
143
155
191

$[-6, 11, 14, 15]$

Definition

Definition

A positive integer *a has a packing*

Symmetric Packings

Definition

A positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

Definition

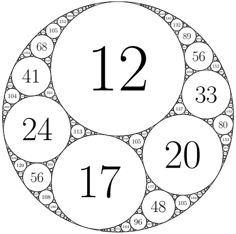
A positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

How many other packings does $a = 7$ have? How many of them are symmetric?

Definition

A positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

How many other packings does $a = 7$ have? How many of them are symmetric?



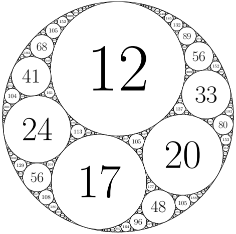
$[-7, 12, 17, 20]$,

Symmetric Packings

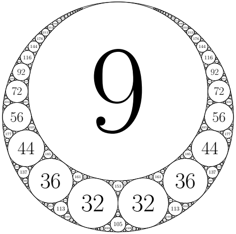
Definition

A positive integer a *has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

How many other packings does $a = 7$ have? How many of them are symmetric?



$[-7, 12, 17, 20]$,

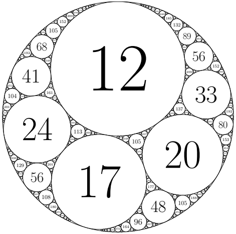


$[-7, 9, 32, 32]$,

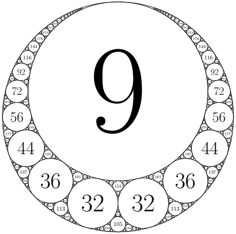
Definition

A positive integer a *has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

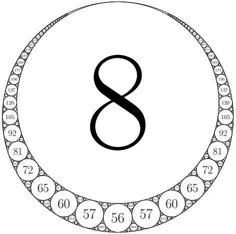
How many other packings does $a = 7$ have? How many of them are symmetric?



$[-7, 12, 17, 20]$,



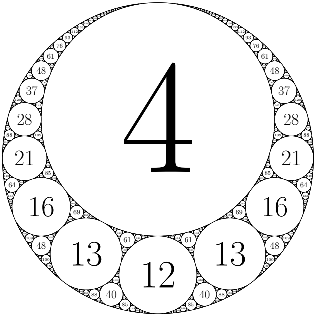
$[-7, 9, 32, 32]$,



$[-7, 8, 56, 57]$.

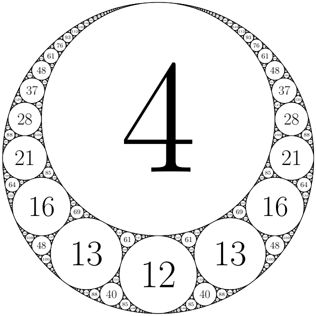
Symmetric Packings

Symmetric Packings

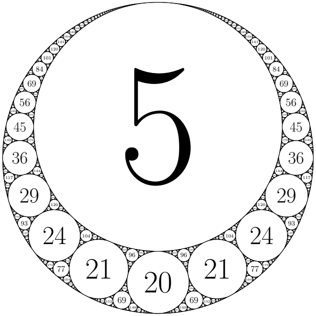


$[-3, 4, 12, 13]$

Symmetric Packings

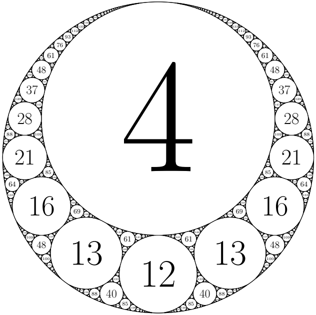


$[-3, 4, 12, 13]$

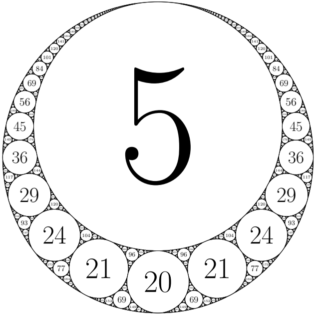


$[-4, 5, 20, 21]$

Symmetric Packings

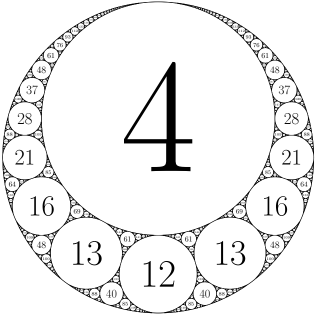


$[-3, 4, 12, 13]$

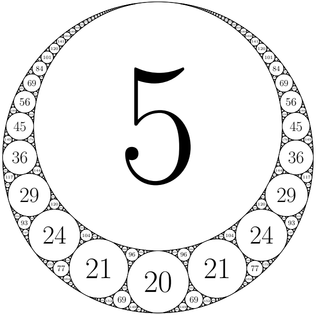


$[-4, 5, 20, 21]$

Definition



$[-3, 4, 12, 13]$

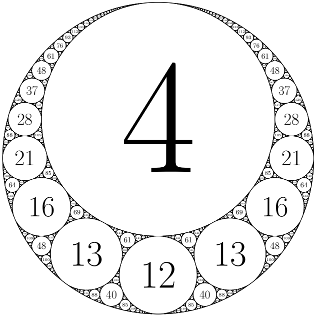


$[-4, 5, 20, 21]$

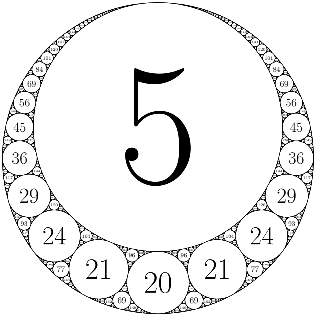
Definition

A *sum-symmetric*

Symmetric Packings



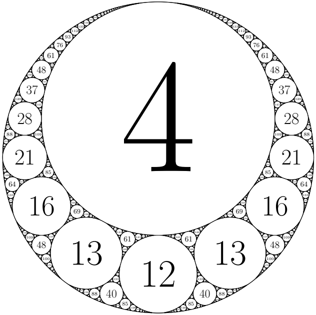
$[-3, 4, 12, 13]$



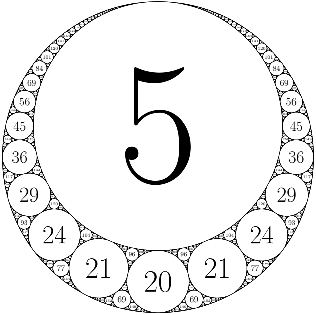
$[-4, 5, 20, 21]$

Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$.



$[-3, 4, 12, 13]$



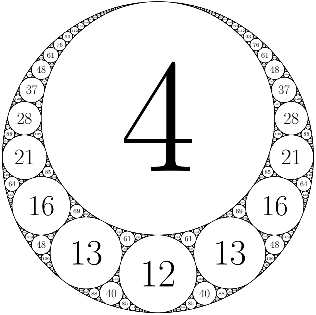
$[-4, 5, 20, 21]$

Definition

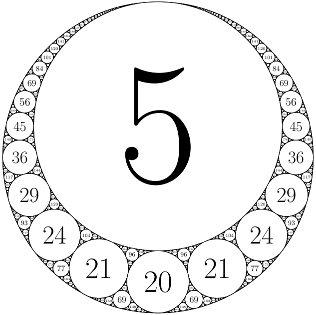
A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$.

$$2(a + b + c) - d = d$$

Symmetric Packings



$[-3, 4, 12, 13]$

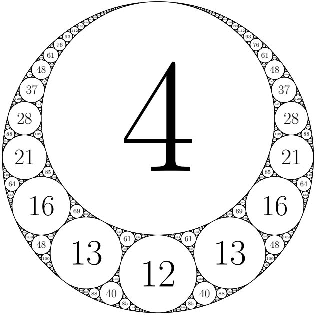


$[-4, 5, 20, 21]$

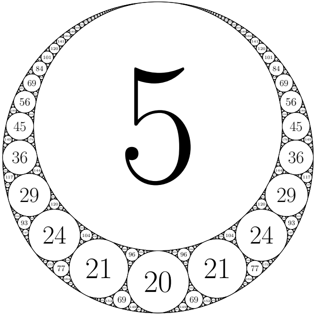
Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a+b+c) - d = d$.

$$2(a+b+c) - d = d \implies 2(a+b+c) = 2d$$



[−3, 4, 12, 13]



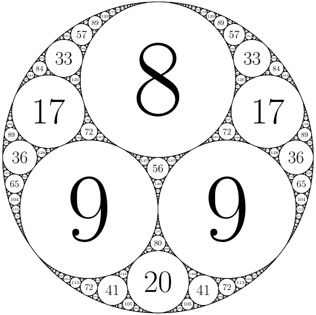
[−4, 5, 20, 21]

Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a+b+c)-d=d$.

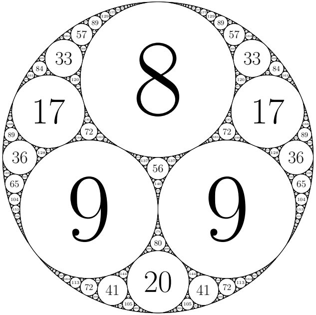
$$2(a+b+c)-d=d \implies 2(a+b+c)=2d \implies a+b+c=d$$

Symmetric Packings

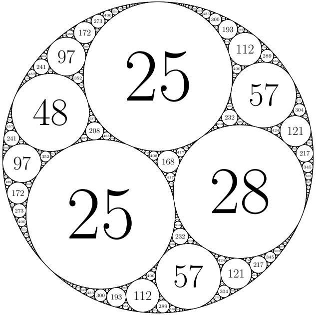


$[-4, 8, 9, 9]$

Symmetric Packings

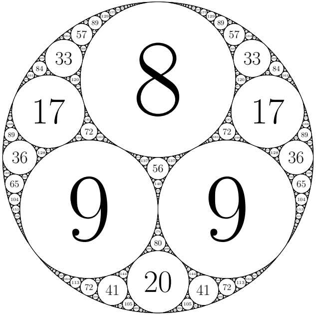


$[-4, 8, 9, 9]$

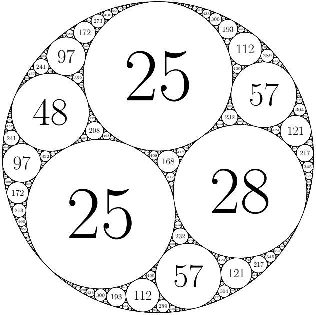


$[-12, 25, 25, 28]$

Symmetric Packings

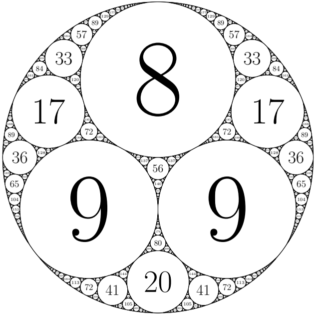


$[-4, 8, 9, 9]$

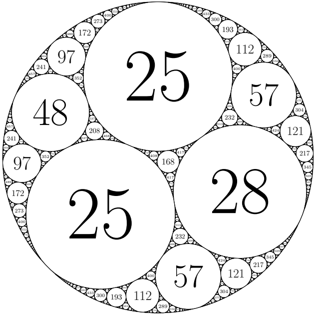


$[-12, 25, 25, 28]$

Definition



$[-4, 8, 9, 9]$

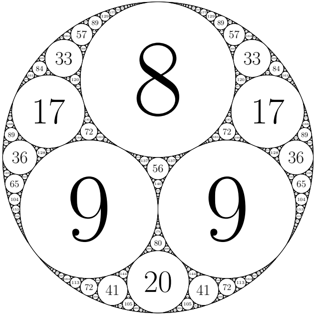


$[-12, 25, 25, 28]$

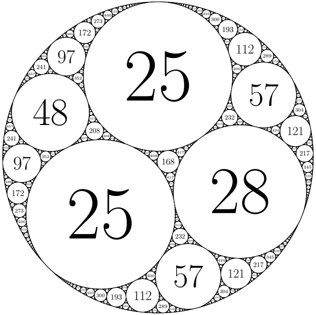
Definition

A *twin-symmetric* quadruple

Symmetric Packings



$[-4, 8, 9, 9]$



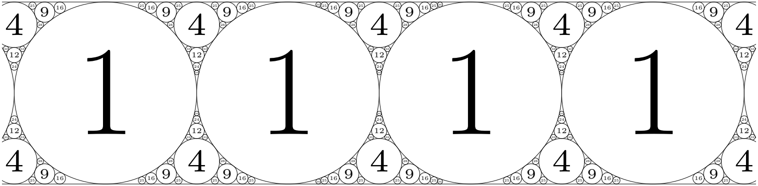
$[-12, 25, 25, 28]$

Definition

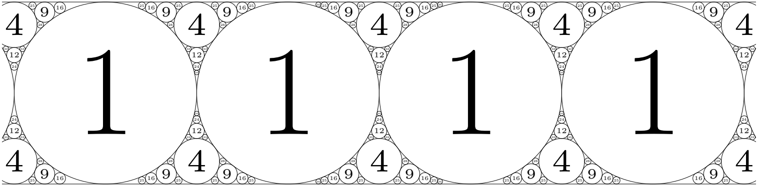
A *twin-symmetric* quadruple is a primitive reduced Descartes quadruple with $c = d$ or $c = b$.

The Two Unusual Symmetric Packings

The Two Unusual Symmetric Packings



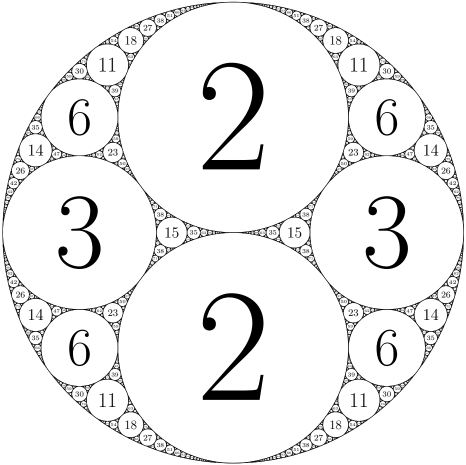
The Two Unusual Symmetric Packings



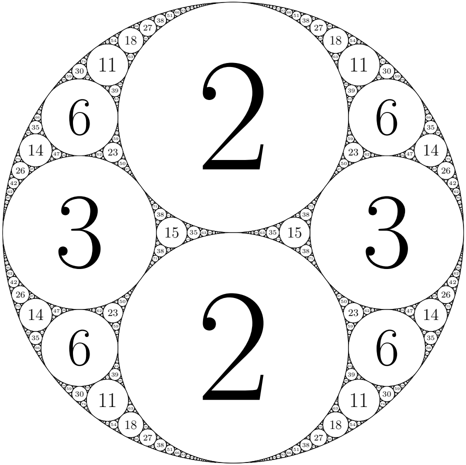
The strip packing: $[0, 0, 1, 1]$

The Two Unusual Symmetric Packings

The Two Unusual Symmetric Packings



The Two Unusual Symmetric Packings



The bug-eye packing: $[-1, 2, 2, 3]$

Symmetric Packings

Proposition (Kertzer, 2024)

Symmetric Packings

Proposition (Kertzer, 2024)

A symmetric packing is either sum-symmetric or twin-symmetric.

Symmetric Packings

Proposition (Kertzer, 2024)

A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition (Kertzer, 2024)

Symmetric Packings

Proposition (Kertzer, 2024)

A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition (Kertzer, 2024)

Only the strip and bug-eye packing are both sum-symmetric and twin-symmetric.

Sum-Symmetric Packings

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$			

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81
$[-21, 30, 70, 79]$	9	49	100

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$d-b$	$d+a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2		3^2		5^2
$[-12, 21, 28, 37]$	3^2		4^2		7^2
$[-18, 22, 99, 103]$	2^2		9^2		11^2
$[-20, 36, 45, 61]$	4^2		5^2		9^2
$[-21, 30, 70, 79]$	3^2		7^2		10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$b+a$	$d-b$	$c+a$	$d+a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2\cdot 3)}_{-6}, \underbrace{2^2+2\cdot 3}_{10}, \underbrace{3^2+2\cdot 3}_{15}, \underbrace{(2+3)^2-2\cdot 3}_{19}\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2\cdot 3)}_{-6},\ \underbrace{2^2 + 2\cdot 3}_{10},\ \underbrace{3^2 + 2\cdot 3}_{15},\ \underbrace{(2+3)^2 - 2\cdot 3}_{19}\right]$$

$$\left[-xy,\ x^2 + xy,\ y^2 + xy,\ (x+y)^2 - xy\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2 \cdot 3)}_{-6},\ \underbrace{2^2 + 2 \cdot 3}_{10},\ \underbrace{3^2 + 2 \cdot 3}_{15},\ \underbrace{(2+3)^2 - 2 \cdot 3}_{19}\right]$$

$$\left[-xy,\ x^2 + xy,\ y^2 + xy,\ (x+y)^2 - xy\right]$$

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$d-b$	$d+a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2 \cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52]$$

$[-a, b, c, d]$	$d-c$	$d-b$	$d+a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2\cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2\cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2\cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2\cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13] \quad (x = 3, y = 1)$$

Sum-Symmetric Packings

Sum-Symmetric Packings

Theorem (Kertzer, 2024)

Sum-Symmetric Packings

Theorem (Kertzer, 2024)

A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

Sum-Symmetric Packings

Theorem (Kertzer, 2024)

A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2-xy\right]$$

Sum-Symmetric Packings

Theorem (Kertzer, 2024)

A sum-symmetric quadruple $[a, b, c, d]$ *is of the form*

$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right]$$

with $\gcd(x, y) = 1$*, and* $x, y \geqslant 0$*.*

The Number of Sum-Symmetric Packings

The Number of Sum-Symmetric Packings

Corollary

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ *sum-symmetric packings, where* $\omega(n)$ *is the number of distinct prime divisors of n.*

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y,

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy but we divide by two to account for symmetry.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y, write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y, so there 2^k total factor pairs xy but we divide by two to account for symmetry. Thus, n has $2^k/2 = 2^{k-1} = 2^{\omega(n)-1}$ sum-symmetric packings. □

Sum-Symmetric packings of 60

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$.

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.

These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$. They are

$$
\begin{aligned}
(1, 60) &\implies [-60, 61, 3660, 3661] \\
(4, 15) &\implies [-60, 76, 285, 301] \\
(3, 20) &\implies [-60, 69, 460, 469] \\
(5, 12) &\implies [-60, 85, 204, 229]
\end{aligned}
$$

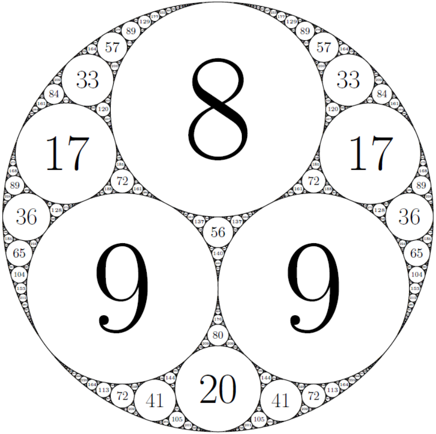
Twin-Symmetric Packings

Twin-Symmetric Packings

Packings where one of the numbers is the same:

Twin-Symmetric Packings

Packings where one of the numbers is the same:



$[-4, 8, 9, 9]$

Twin-Symmetric Packings

Clyde Kertzer

Twin-Symmetric Packings

-2	none

-2	none
-3	$[-3, 5, 8, 8]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$
-16	$[-16, 20, 81, 81]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$
-16	$[-16, 20, 81, 81]$

Twin-Symmetric Packings

Twin-Symmetric Packings

Clyde Kertzer

Over the summer:

Over the summer:

Theorem

Twin-Symmetric Packings

Over the summer:

Theorem

All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

$$\left[-x,\ x+2y^2,\ 2\left(\frac{x+y^2}{2y}\right)^2,\ 2\left(\frac{x+y^2}{2y}\right)^2\right] \quad y \text{ odd}$$

Twin-Symmetric Packings

Over the summer:

Theorem

All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

$$\left[-x,\ x+2y^2,\ 2\left(\frac{x+y^2}{2y}\right)^2,\ 2\left(\frac{x+y^2}{2y}\right)^2\right] \quad y \text{ odd}$$

Not ideal, not in terms of factorization.

Twin-Symmetric Packings

Improved to:

Improved to:

Theorem

Twin-Symmetric Packings

Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \dfrac{(x+y)^2}{2},\ \dfrac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x > y \\ \\ \\ \end{cases}$$

Twin-Symmetric Packings

Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \dfrac{(x+y)^2}{2},\ \dfrac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x>y \\ \left[-xy,\ xy+4y^2,\ \left(\dfrac{x}{2}+y\right)^2,\ \left(\dfrac{x}{2}+y\right)^2\right] & 4 \mid x, \quad x>2y \\ \end{cases}$$

Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \dfrac{(x+y)^2}{2},\ \dfrac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x>y \\ \left[-xy,\ xy+4y^2,\ \left(\dfrac{x}{2}+y\right)^2,\ \left(\dfrac{x}{2}+y\right)^2\right] & 4 \mid x, \quad x>2y \\ \left[-xy,\ xy+x^2,\ \left(\dfrac{x}{2}+y\right)^2,\ \left(\dfrac{x}{2}+y\right)^2\right] & 4 \mid x, \quad x<2y \end{cases}$$

with $\gcd(x,y)=1$.

Twin-Symmetric Packings

Further improved to:

Twin-Symmetric Packings

Further improved to:

Theorem (Kertzer, 2024)

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \tfrac{1}{2}(x+y)^2,\ \tfrac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Twin-Symmetric Packings

Further improved to:

Theorem (Kertzer, 2024)

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Ex: $x = 3$, $y = 2$

Further improved to:

Theorem (Kertzer, 2024)

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Ex: $x = 3,\ y = 2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Further improved to:

Theorem (Kertzer, 2024)

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

Ex: $x = 3,\ y = 2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Why won't $x = 1, y = 3$ work?

Further improved to:

Theorem (Kertzer, 2024)

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x,y \text{ odd} \quad x>y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x>y \end{cases}$$

with $\gcd(x,y)=1$ *and* $x,y \geqslant 0$.

Ex: $x=3,\ y=2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Why won't $x=1, y=3$ work? Let's try:

Further improved to:

Theorem (Kertzer, 2024)

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x,y \textit{ odd} \quad x>y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \textit{ even} \quad x>y \end{cases}$$

with $\gcd(x,y)=1$ *and* $x,y \geqslant 0$.

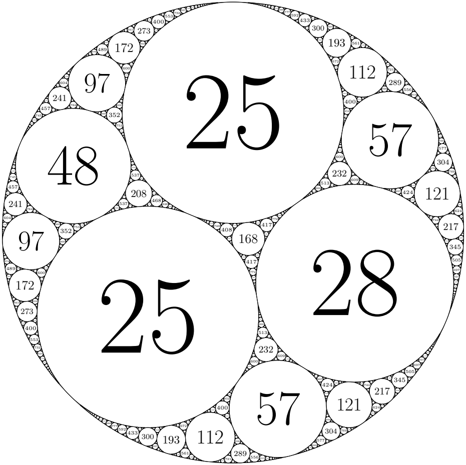
Ex: $x=3,\ y=2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

Why won't $x=1, y=3$ work? Let's try:

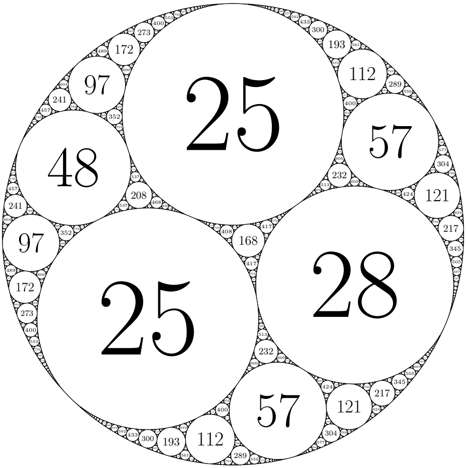
$$[-3, 3+2(3)^2, 5^2, 5^2] \implies [-12, 48, 25, 25]$$

Twin-symmetric Packings


25
25
28
48
57
57
97
97
112
112
121
121
168
172
172
193
193
208
217
217
232
232
241
241
273
273
289
289
300
300
304
304
345
345
352
352
400
400
408
417
417
424
424
433
433
457
457
489
489
505
505
513
513
537
537
553
553
561
561
577
577
592
592

$[-12, 48, 25, 25]$

Twin-symmetric Packings



$$[-12, 48, 25, 25] \implies [-12, 28, 25, 25]$$

The Number of Twin-symmetric Packings

The Number of Twin-symmetric Packings

We define δ_n as

The Number of Twin-symmetric Packings

We define δ_n as

$$\delta_n = \begin{cases} 1 & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

We define δ_n as

$$\delta_n = \begin{cases} 1 & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary

The Number of Twin-symmetric Packings

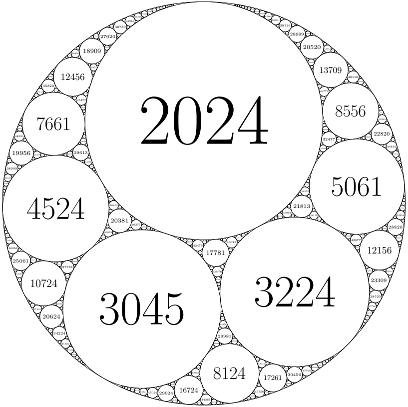
We define δ_n as

$$\delta_n = \begin{cases} 1 & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary

A natural number n has $(1 - \delta_n) \cdot 2^{\omega(n)-1}$ twin-symmetric packings where $\omega(n)$ is the number of distinct prime divisors of n.

Thank You!



Images generated using James Rickards' Code.

Circles checklist:

-7 (outer ring), 8, 9, 10, 11, 12, 15, 18, 19, 20, 22, 24, 25, 28, 32 (x2), 33, 35, 38, 41, 44, 48, 56 (x2), 57, 64, 72, 80, 88, 96, 105 (x3)

References

Tabes Bridges, Warren Tai, and Karol Koziol. A classification of integral apollonian circle packings. https://www.math.columbia.edu/-staff/RTGPapers/IntegralClassification.pdf, 2011.

Duncan Buell. Binary Quadratic Forms. Springer-Verlag, Berlin, 1989.

David A. Cox. Primes of the form $x^2 + ny^2$. Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, N.J, second edition, 2013. Fermat, class field theory, and complex multiplication.

Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian Circle Packings: Number Theory. J. Number Theory, 100(1):1-45, 2003.

James Rickards. Apollonian. https://github.com/JamesRickards-Canada/Apollonian, 2023.