Packing Problems & Number Theory

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University of Colorado Boulder

December 12, 2024

A classic game

A classic game

A guessing jar:

A classic game

A guessing jar: How many marbles?

A classic game

A guessing jar: How many marbles? 223!

A classic game

A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?

A classic game

A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?
2. Given 223 marbles, what's the smallest container holding them all?

A classic game

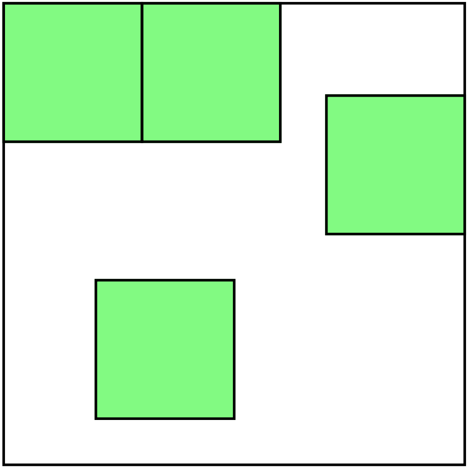
A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?
2. Given 223 marbles, what's the smallest container holding them all?

We need to simplify the problem...

Square Packing

Square Packing



Square Packing

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What's the smallest square we can fit 4 squares inside of?

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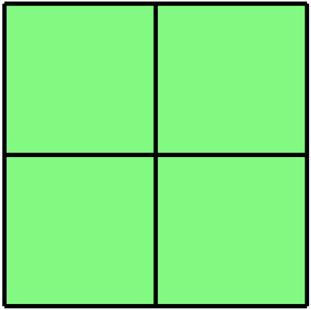

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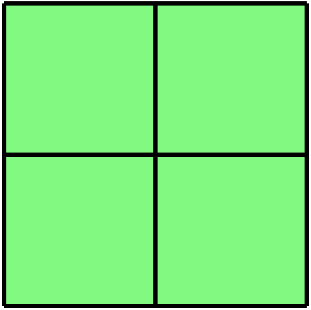

Side length: 2

What's the smallest square we can fit 9 squares inside of?

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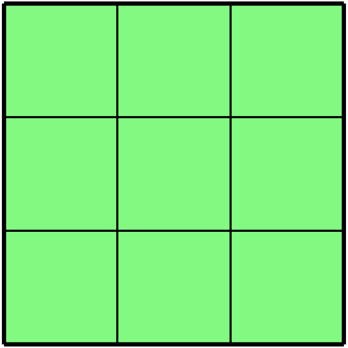

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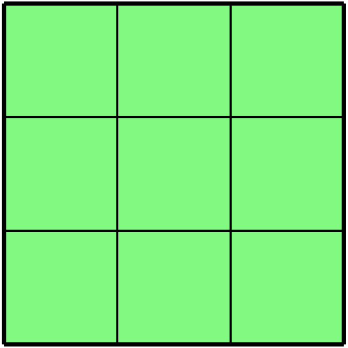

Side length: 3

Square Packing

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What about 5 squares?

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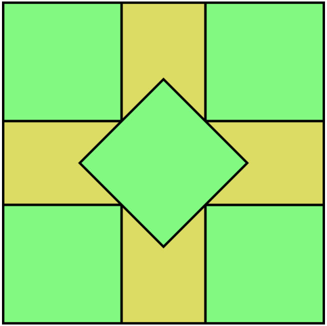

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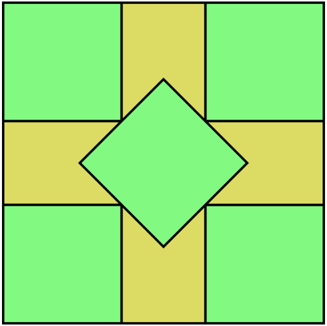

Side length:

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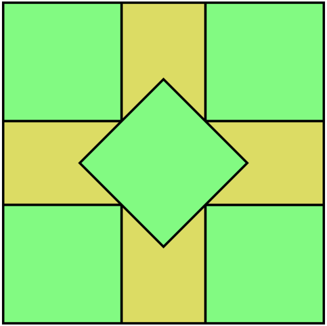

Side length: $\approx 2.707\ldots$

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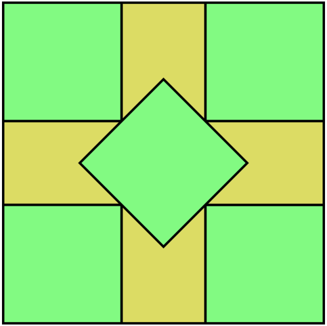

Side length: $\approx 2.707\ldots = 2 + \sqrt{2}/2$

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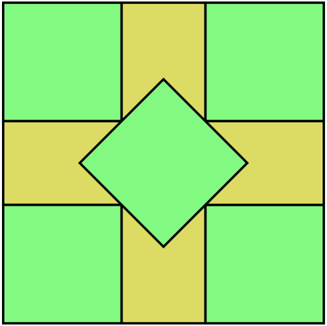

Side length: $\approx 2.707\ldots = 2 + \sqrt{2}/2$

Can we use this packing to find the optimal packing of 10 squares?

Square Packing

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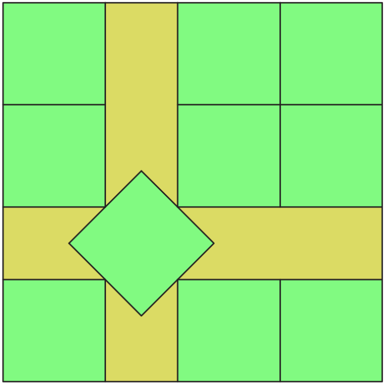

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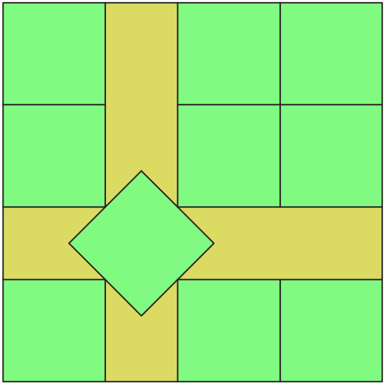

Side length: $\approx 3.707\ldots = 3 + \frac{\sqrt{2}}{2}$

Square Packing

What about 11 squares?

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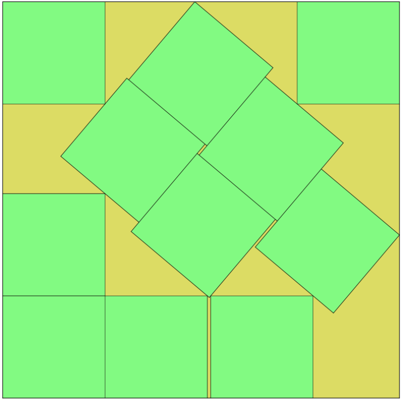
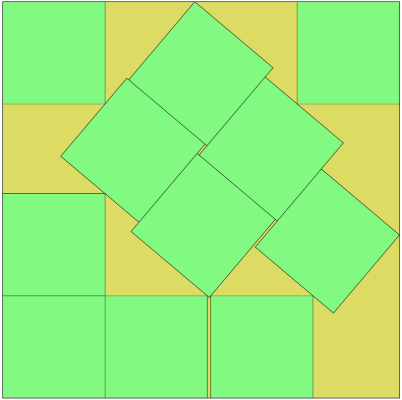


What about 11 squares?



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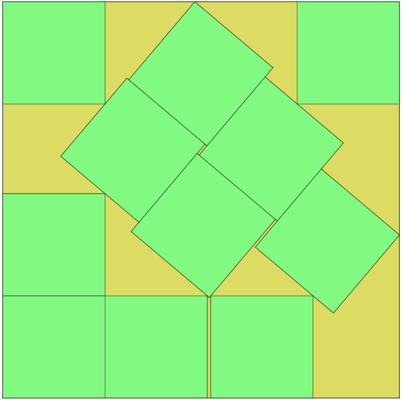

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Is this the best possible packing?

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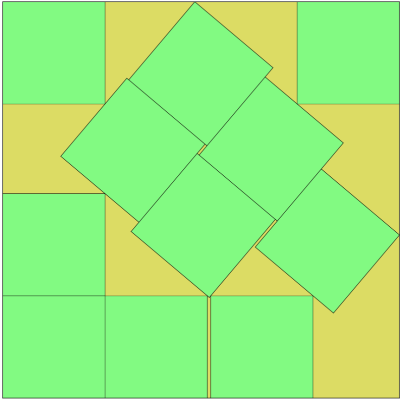

Side length: $\approx 3.877\ldots$

Is this the best possible packing? Mathematicians still don't know...

Circle Packing

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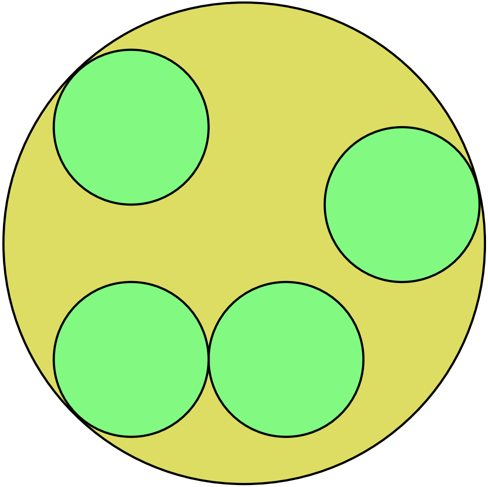

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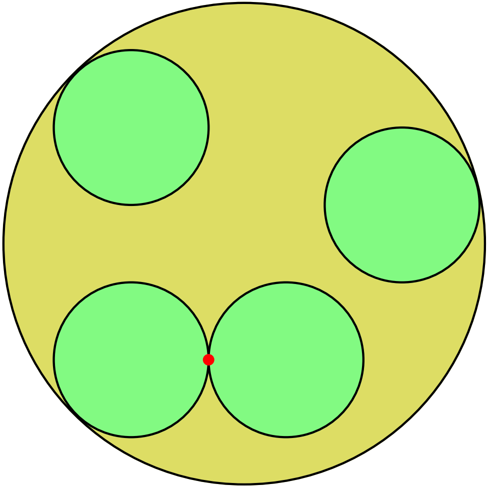
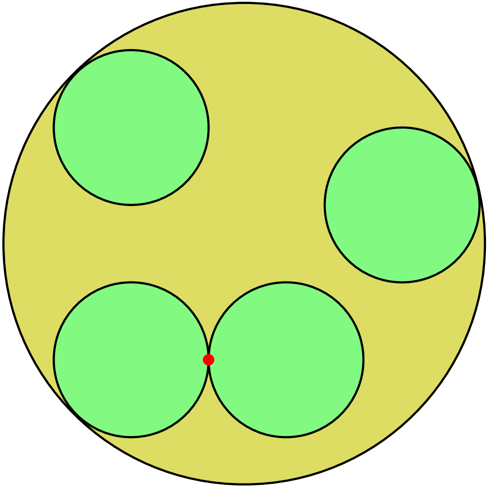
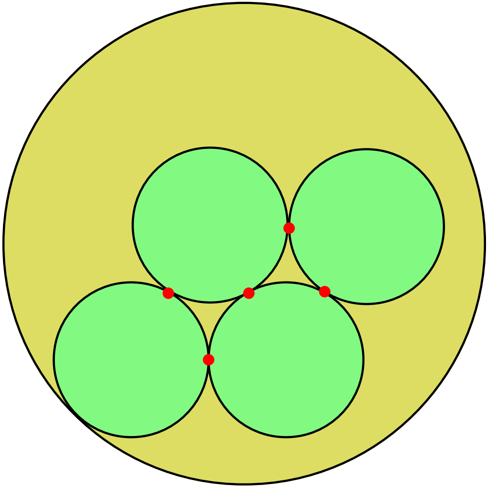
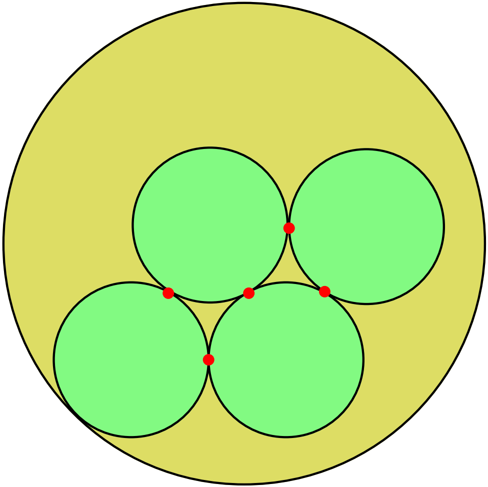




Try it yourself: Can you find an arrangement of 4 circles such that each one is tangent to the other?

Circle Packing





What if the circles aren't all the same size...

Circle Packing

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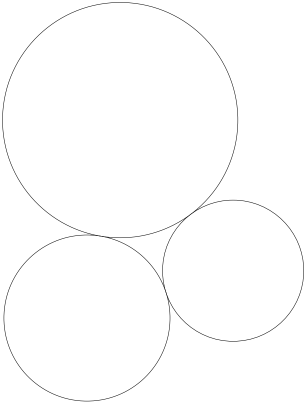

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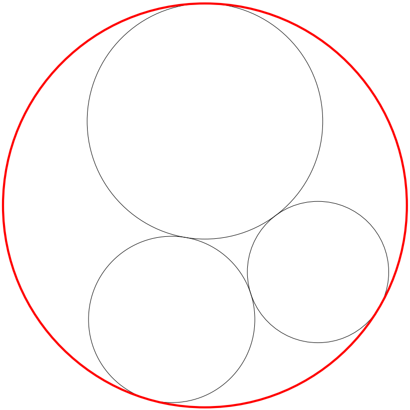

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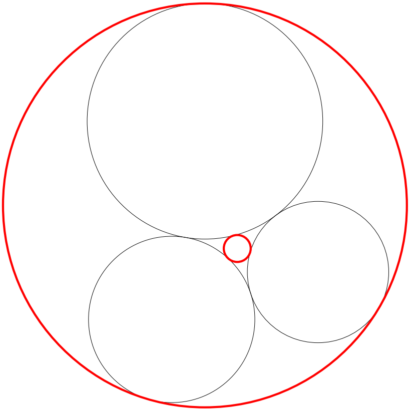

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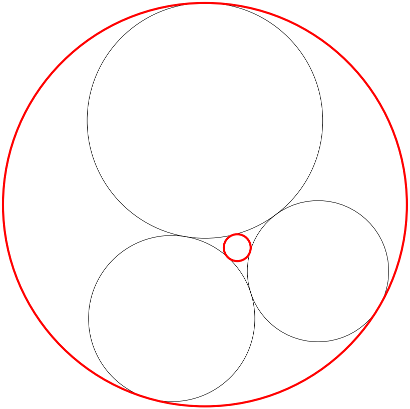

Definition

A set of four mutually tangent circles is called a *Descartes Quadruple*

Circle Packing: Curvature

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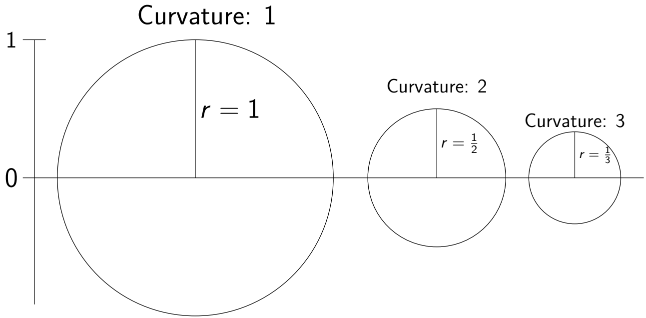

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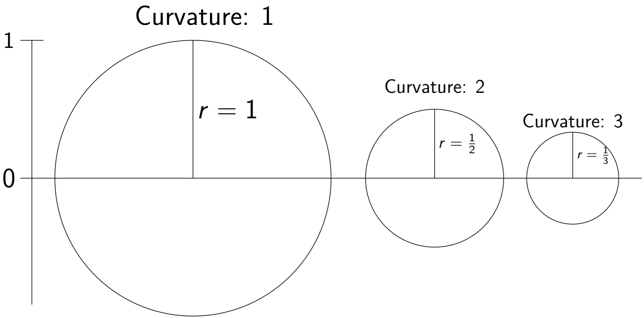

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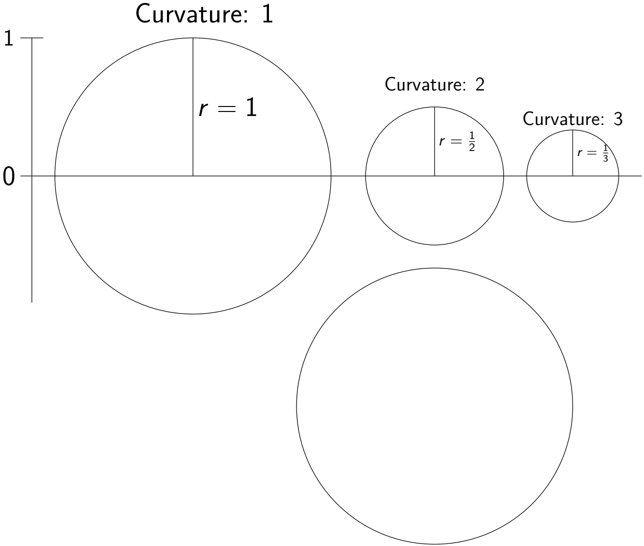

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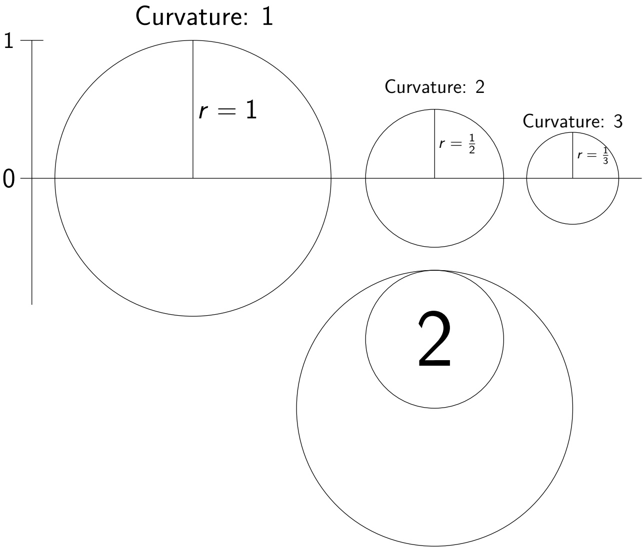

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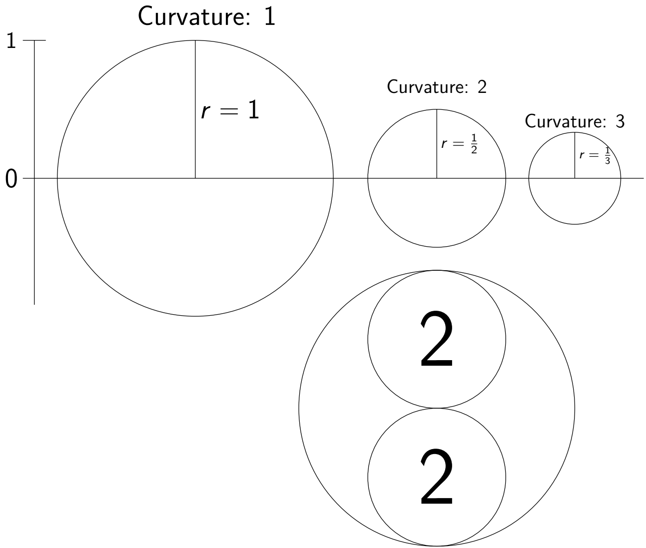

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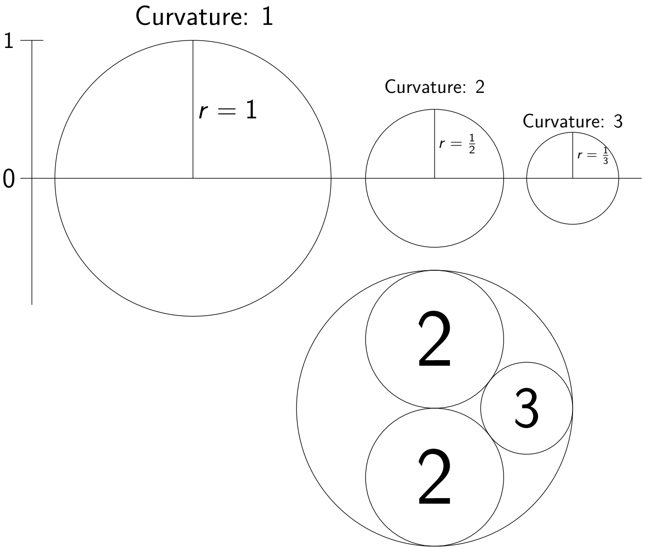

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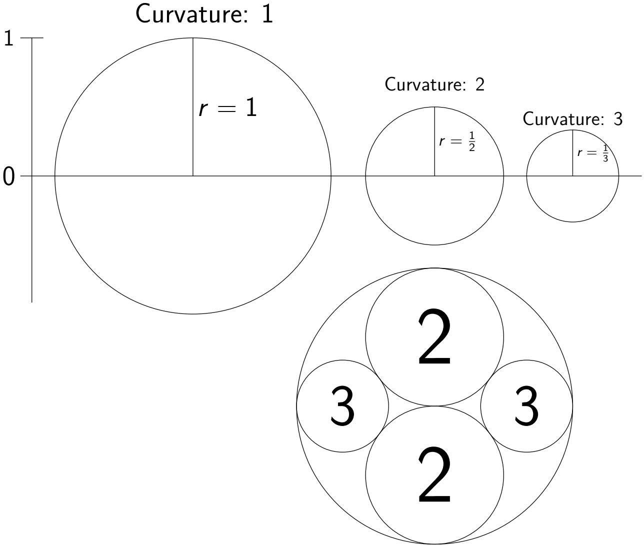

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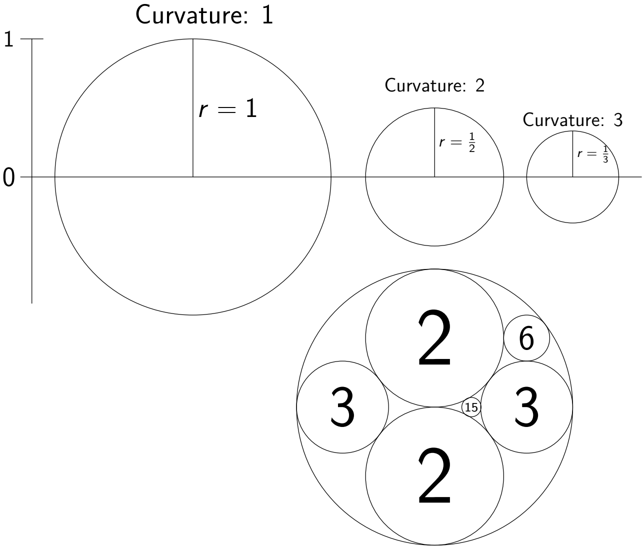

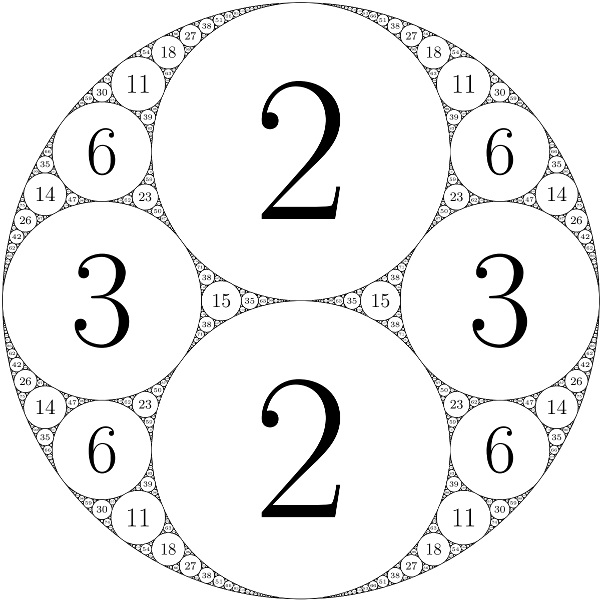
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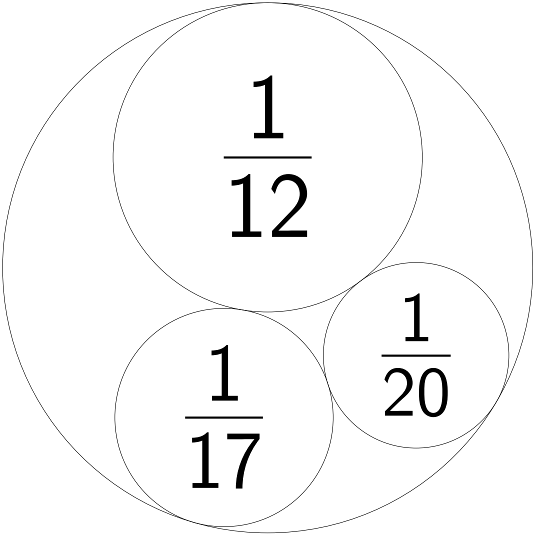
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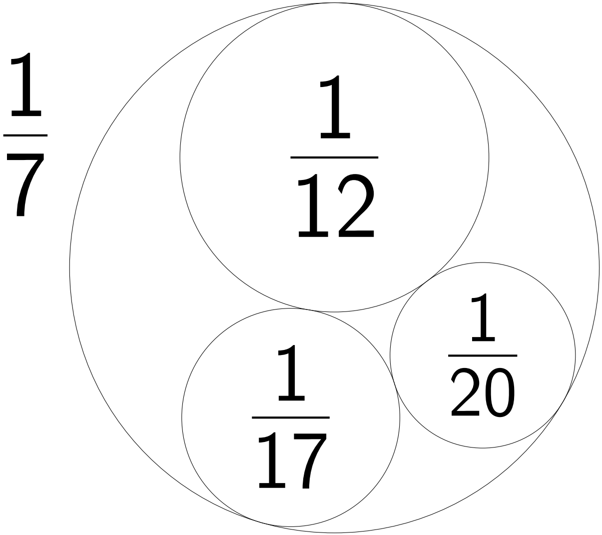
Circle Packing



Circle Packing

Clyde Kertzer

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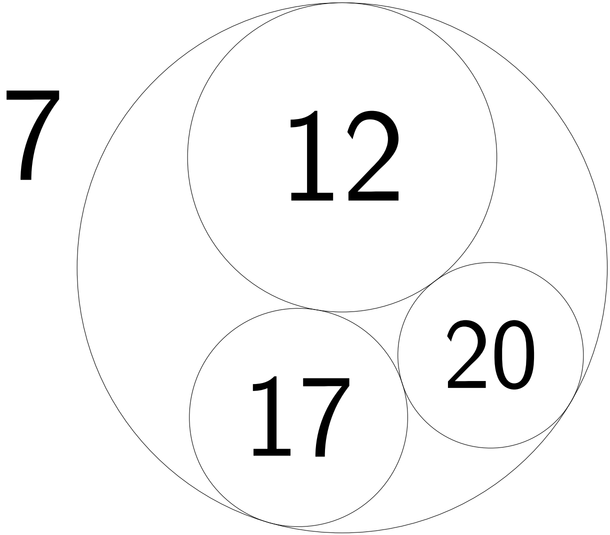

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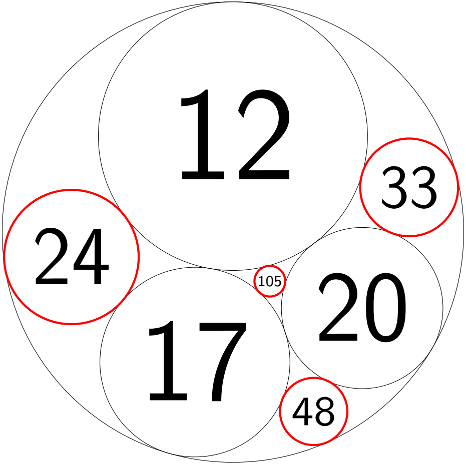

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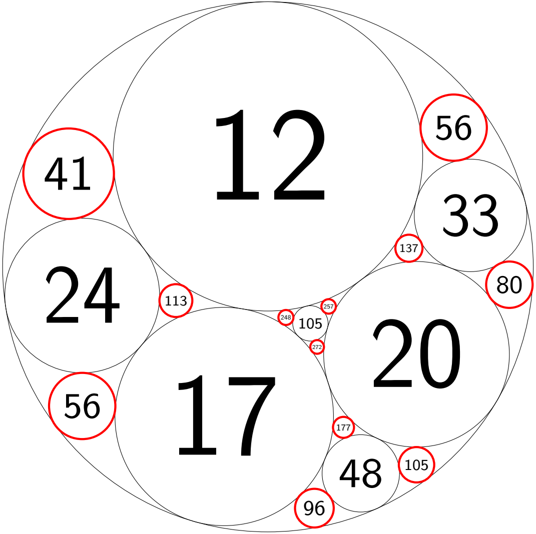

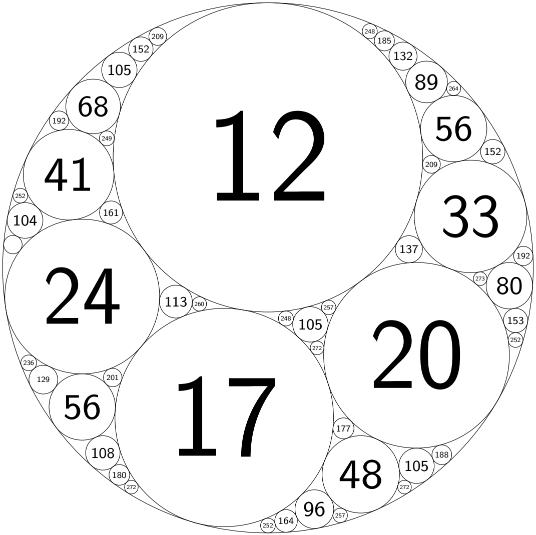
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The Descartes Equation

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The *curvature* of a circle with radius r is defined to be $1/r$.

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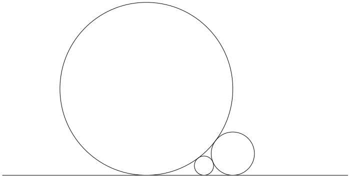

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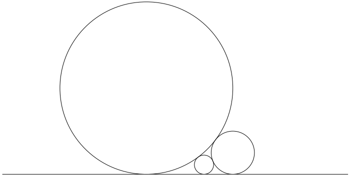

Circle with infinite radius (Curvature 0)

The Descartes Equation

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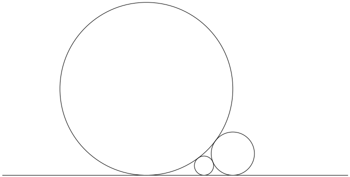

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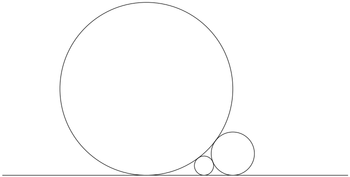

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If four mutually tangent circles have curvatures a, b, c, d then

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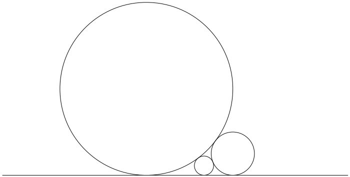

Circle with infinite radius (Curvature 0)

Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

The Descartes Equation

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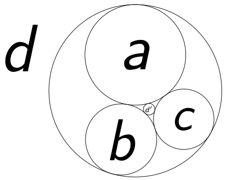
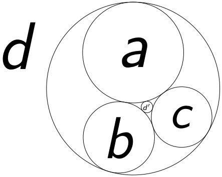
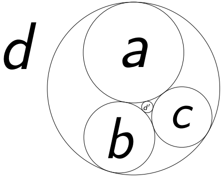


The Descartes Equation



Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures



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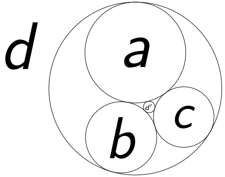
$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Descartes Equation



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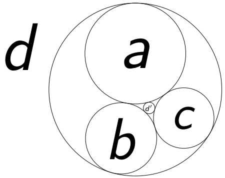
The Key Relation

The Descartes Equation



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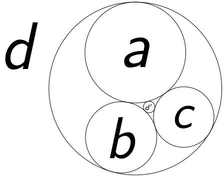
The Key Relation

$d + d' = 2(a + b + c)$

The Descartes Equation



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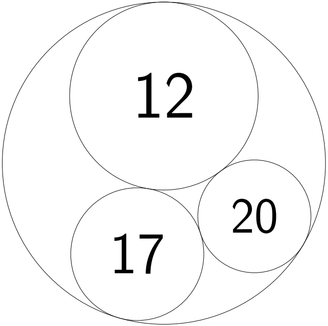
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

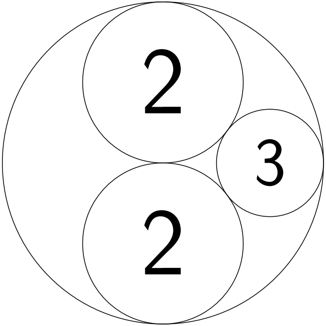
The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

The Descartes Equation

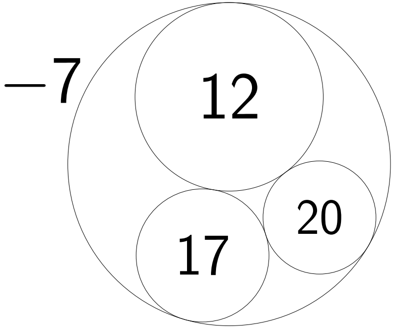


$[-7, 12, 17, 20]$



$[-1, 2, 2, 3]$

The Descartes Equation



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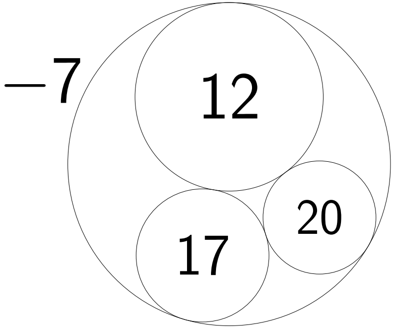

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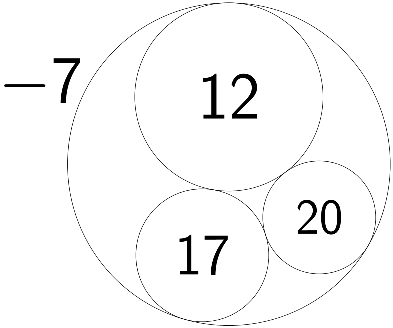
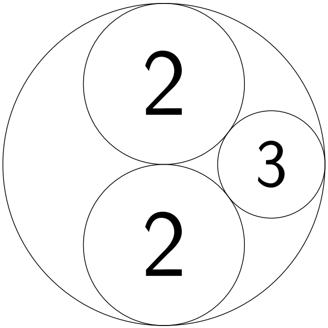
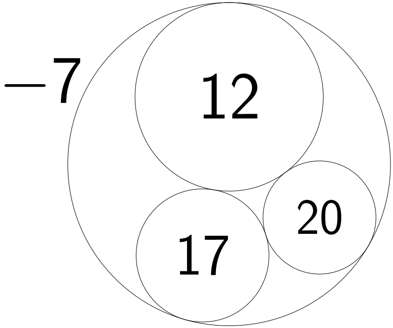
The Descartes Equation



$[-7, 12, 17, 20]$



The Descartes Equation



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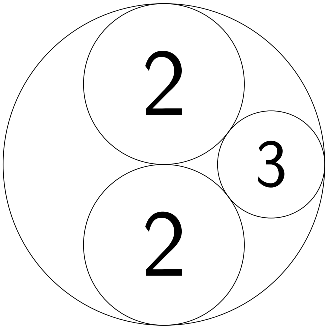

$[-1, 2, 2, 3]$

The Descartes Equation

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Proof.

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The quadratic formula gives

$$d=(a+b+c) \pm \frac{\sqrt{4(a+b+c)^2-4\left(a^2+b^2+c^2-2ab-2bc-2ac\right)}}{2}$$
$$=a+b+c\pm 2\sqrt{ab+bc+ca}.$$

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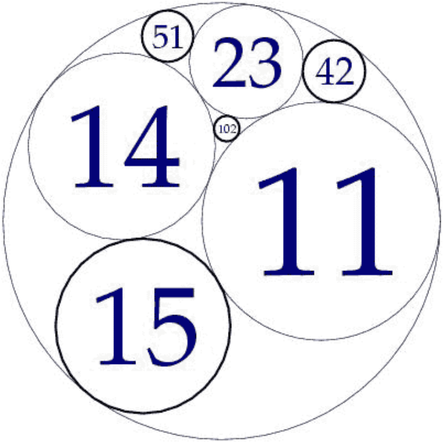
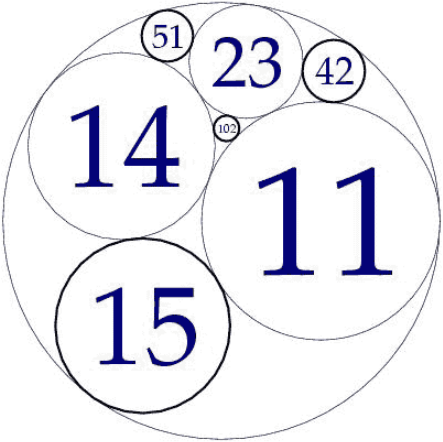
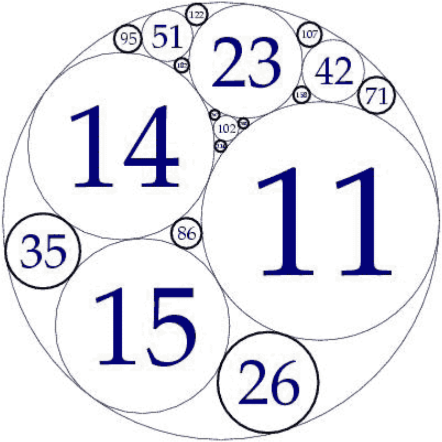
Thus, there are two options for d. Their sum is $2(a+b+c)$. □



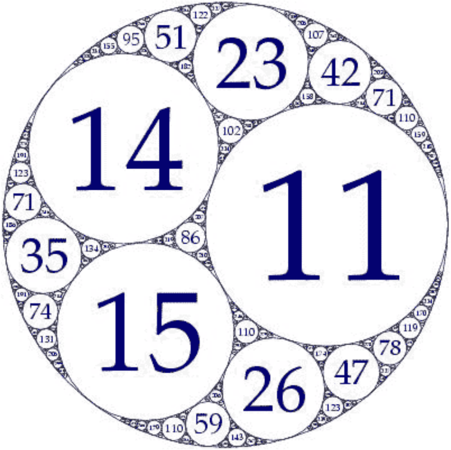
$[-6, 11, 14, 23]$



$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$



$[-6, 11, 14, 15]$


11
14
15
23
26
35
42
47
51
59
71
71
74
78
86
95
102
107
110
110
110
119
122
123
123
131
134
143

$[-6, 11, 14, 15]$

Definition

Apollonian Circle Packings

Definition

A positive integer *a has a packing*

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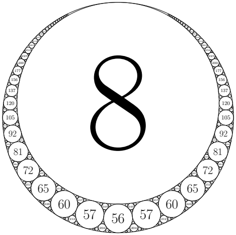

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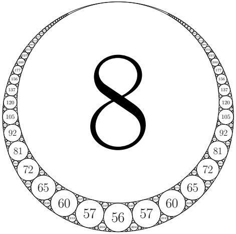

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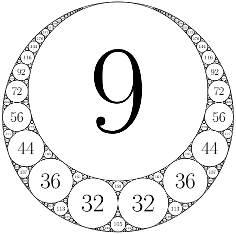

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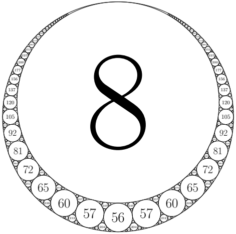

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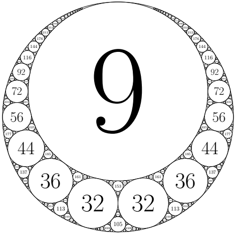

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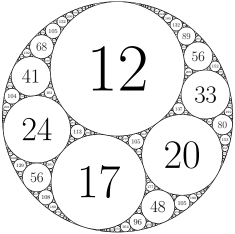
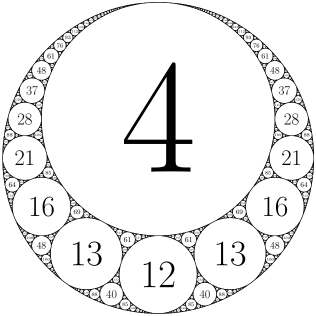


$[-7, 12, 17, 20]$.

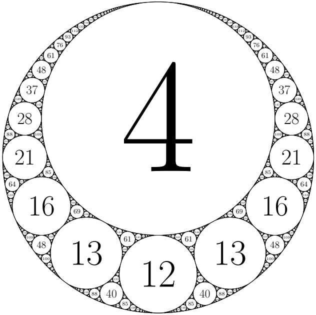
Symmetric Packings

Symmetric Packings

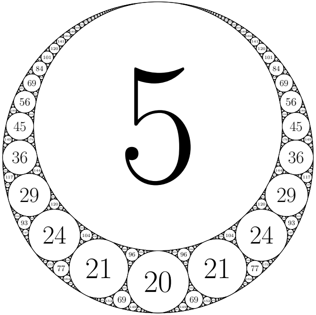


[−3, 4, 12, 13]

Symmetric Packings

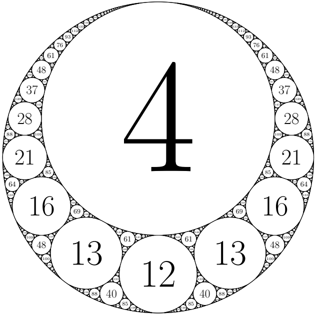


[−3, 4, 12, 13]



[−4, 5, 20, 21]

Symmetric Packings



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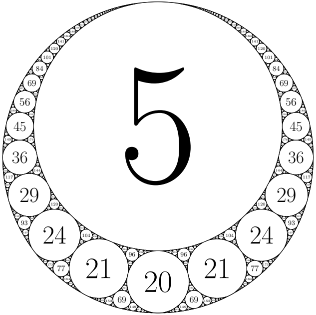

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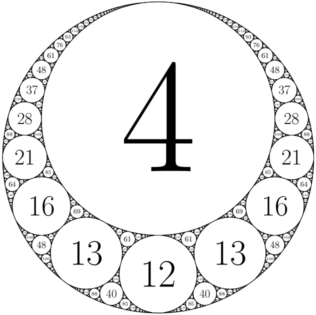

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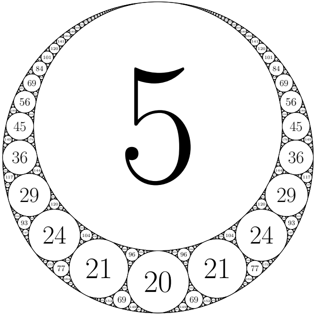
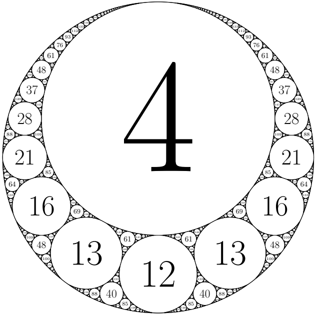


[−4, 5, 20, 21]

Definition

A *sum-symmetric*

Symmetric Packings



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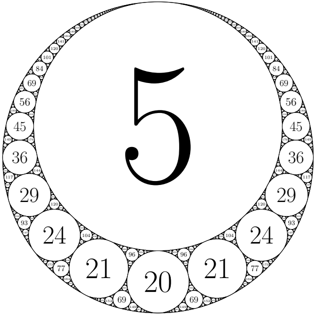

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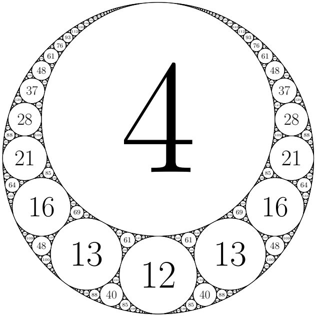
Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$.

Symmetric Packings



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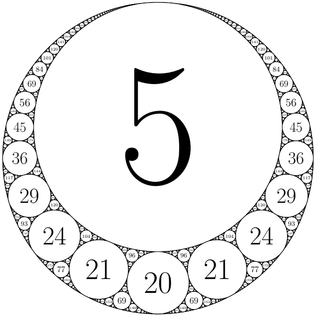

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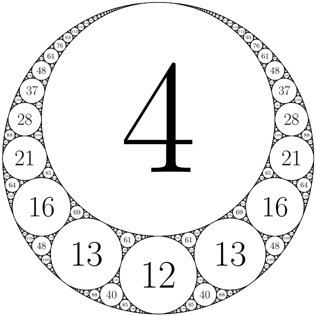

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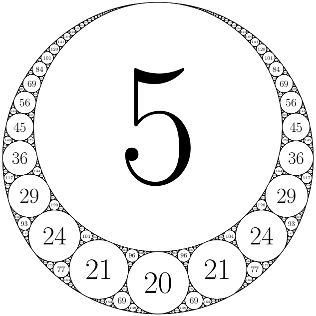

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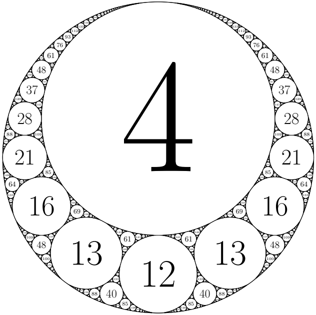
$$2(a+b+c)-d=d \implies 2(a+b+c)=2d$$

Symmetric Packings



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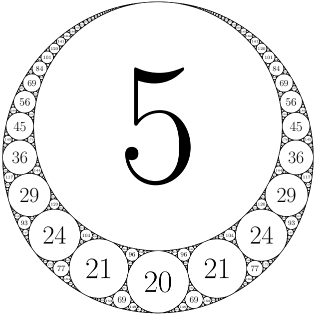

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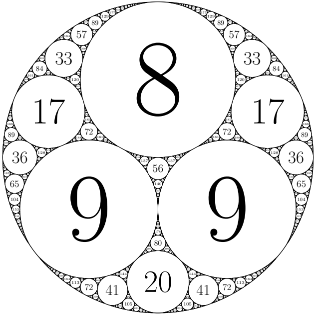
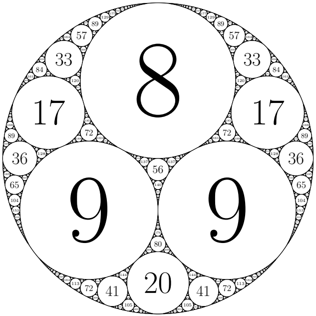
Symmetric Packings

Symmetric Packings



[−4, 8, 9, 9]

Symmetric Packings



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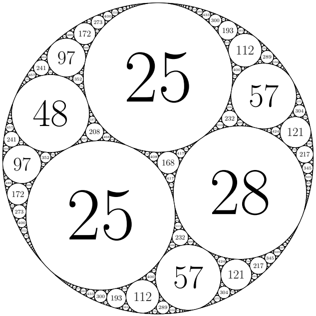
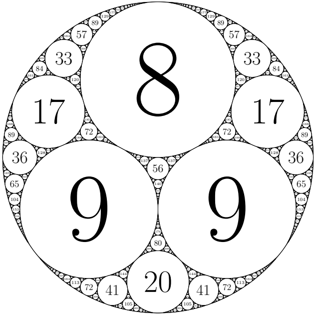


[−12, 25, 25, 28]

Symmetric Packings



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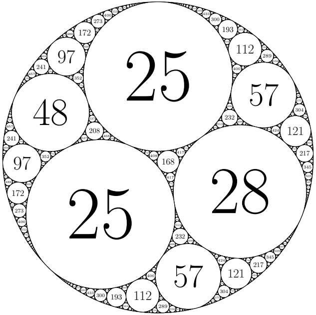

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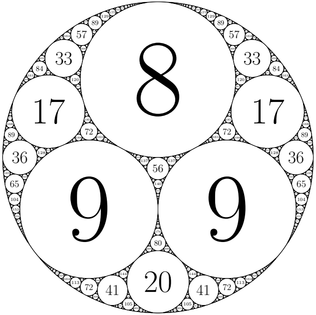
Symmetric Packings



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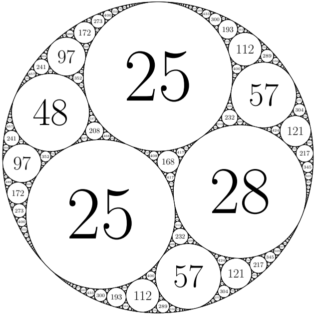
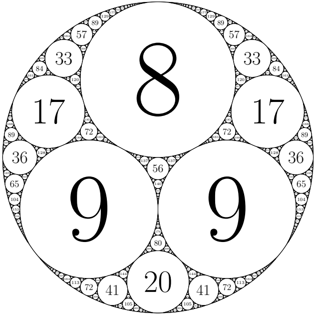


$[-12, 25, 25, 28]$

Definition

A *twin-symmetric* quadruple

Symmetric Packings



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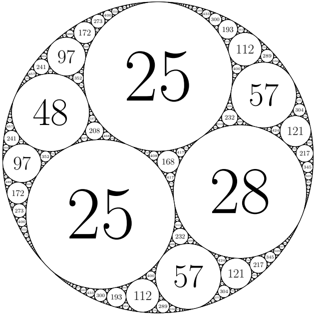

$[-12, 25, 25, 28]$

Definition

A *twin-symmetric* quadruple is a primitive reduced Descartes quadruple with $c = d$ or $c = b$.

The Two Unusual Symmetric Packings

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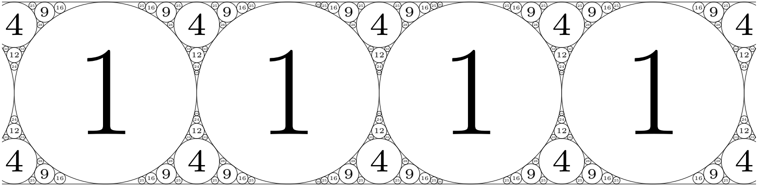
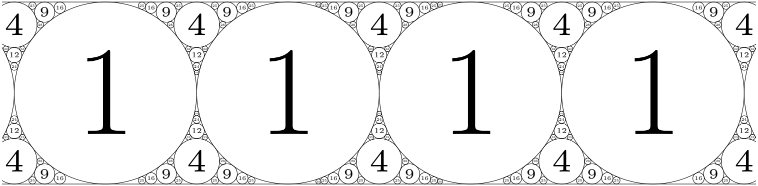


The Two Unusual Symmetric Packings



The strip packing: $[0, 0, 1, 1]$

The Two Unusual Symmetric Packings

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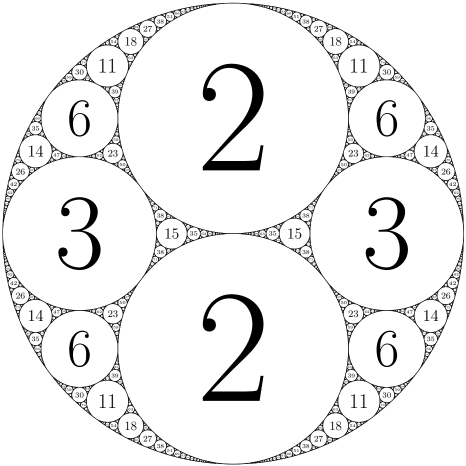
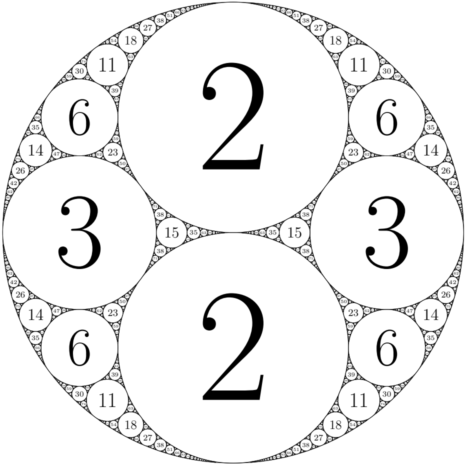


The Two Unusual Symmetric Packings



The bug-eye packing: $[-1, 2, 2, 3]$

Symmetric Packings

Proposition

Symmetric Packings

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

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A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition

Only the strip and bug-eye packing are both sum-symmetric and twin-symmetric.

Sum-Symmetric Packings

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$			

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$d-b$	$d+a$
$[-6, 10, 15, 19]$	4	9	25

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
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$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81
$[-21, 30, 70, 79]$	9	49	100

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2		3^2		5^2
$[-12, 21, 28, 37]$	3^2		4^2		7^2
$[-18, 22, 99, 103]$	2^2		9^2		11^2
$[-20, 36, 45, 61]$	4^2		5^2		9^2
$[-21, 30, 70, 79]$	3^2		7^2		10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d-c$	$b+a$	$d-b$	$c+a$	$d+a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2\cdot 3)}_{-6},\ \underbrace{2^2+2\cdot 3}_{10},\ \underbrace{3^2+2\cdot 3}_{15},\ \underbrace{(2+3)^2-2\cdot 3}_{19}\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
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Given the factorization of a, we can find the entire quadruple!

$$\left[\underbrace{-(2\cdot 3)}_{-6},\ \underbrace{2^2+2\cdot 3}_{10},\ \underbrace{3^2+2\cdot 3}_{15},\ \underbrace{(2+3)^2-2\cdot 3}_{19}\right]$$

$$\left[-xy,\ x^2+xy,\ y^2+xy,\ (x+y)^2-xy\right]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
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Try with $12 = 6 \cdot 2$:

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

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Sum-Symmetric Packings

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$$[-12,\ 16,\ 48,\ 52]$$

Sum-Symmetric Packings

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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2\cdot 6,\ 2(2+6),\ 6(2+6),\ (2+6)^2 - 2\cdot 6\right] =$$

$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13] \quad (x = 3, y = 1)$$

Sum-Symmetric Packings

Theorem

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with $\gcd(x, y) = 1$*, and* $x, y \geqslant 0$.

The Number of Sum-Symmetric Packings

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A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n.

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Sum-Symmetric packings of 60

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Write $60 = 2^2 \cdot 3 \cdot 5$,

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Sum-Symmetric packings of 60

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Sum-Symmetric packings of 60

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These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$. They are

$$\begin{aligned} (1, 60) &\implies [-60, 61, 3660, 3661] \\ (4, 15) &\implies [-60, 76, 285, 301] \\ (3, 20) &\implies [-60, 69, 460, 469] \\ (5, 12) &\implies [-60, 85, 204, 229] \end{aligned}$$

Twin-Symmetric Packings

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Packings where one of the numbers is the same:

Twin-Symmetric Packings

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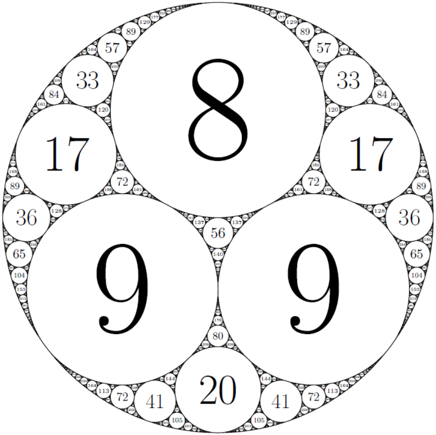

$[-4, 8, 9, 9]$

-2	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$
-16	$[-16, 20, 81, 81]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
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Twin-Symmetric Packings

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Over the summer:

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Over the summer:

Theorem

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Theorem

All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

$$\left[-x,\ x+2y^2,\ 2\left(\frac{x+y^2}{2y}\right)^2,\ 2\left(\frac{x+y^2}{2y}\right)^2\right] \quad y \text{ odd}$$

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Not ideal, not in terms of factorization.

Twin-Symmetric Packings

Improved to:

Improved to:

Theorem

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Theorem

A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \dfrac{(x+y)^2}{2},\ \dfrac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x>y \\ & \end{cases}$$

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with $\gcd(x,y)=1$.

Twin-Symmetric Packings

Further improved to:

Twin-Symmetric Packings

Clyde Kertzer

Further improved to:

Theorem

A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{1}{2}(x+y)^2,\ \frac{1}{2}(x+y)^2\right] & x, y \text{ odd} \quad x > y \\ \left[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2\right] & xy \text{ even} \quad x > y \end{cases}$$

with $\gcd(x, y) = 1$ *and* $x, y \geqslant 0$.

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Ex: $x=3$, $y=2$

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Ex: $x = 3,\ y = 2$:

$$[-12, 12+16, 5^2, 5^2] \implies [-12, 28, 25, 25]$$

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Why won't $x = 1, y = 3$ work?

Further improved to:

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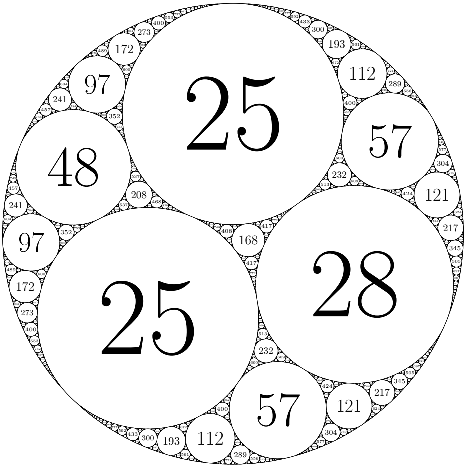
Why won't $x = 1, y = 3$ work? Let's try:

$$[-3, 3+2(3)^2, 5^2, 5^2] \implies [-12, 48, 25, 25]$$

Twin-symmetric Packings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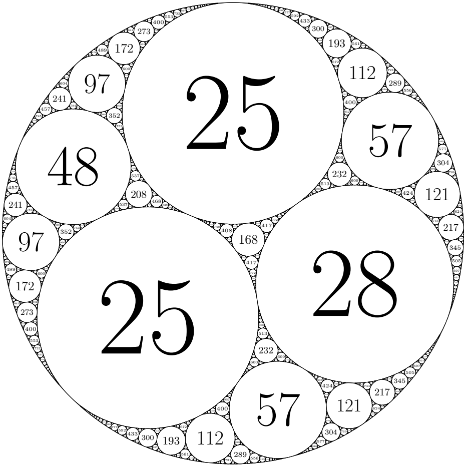
345
345
352
352

$[-12, 48, 25, 25]$

Twin-symmetric Packings



$$[-12, 48, 25, 25] \implies [-12, 28, 25, 25]$$

The Number of Twin-symmetric Packings

We define δ_n as

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$$\delta_n = \begin{cases} 1 & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

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Corollary

The Number of Twin-symmetric Packings

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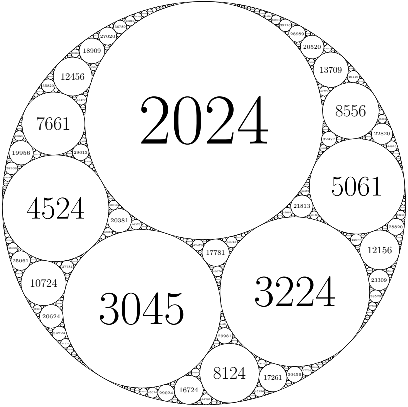
Corollary

A natural number n has $(1 - \delta_n) \cdot 2^{\omega(n)-1}$ twin-symmetric packings where $\omega(n)$ is the number of distinct prime divisors of n.

Thank You!



Images generated using James Rickards' Code.

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