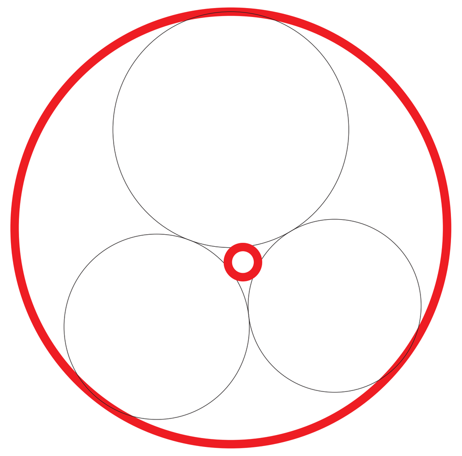
CU Mathematics Lab

THE LOCAL-GLOBAL CONJECTURE IS FALSE

Clyde Kertzer

With Summer Haag, Katherine E. Stange, and James Rickards

APOLLONIAN CIRCLE PACKINGS

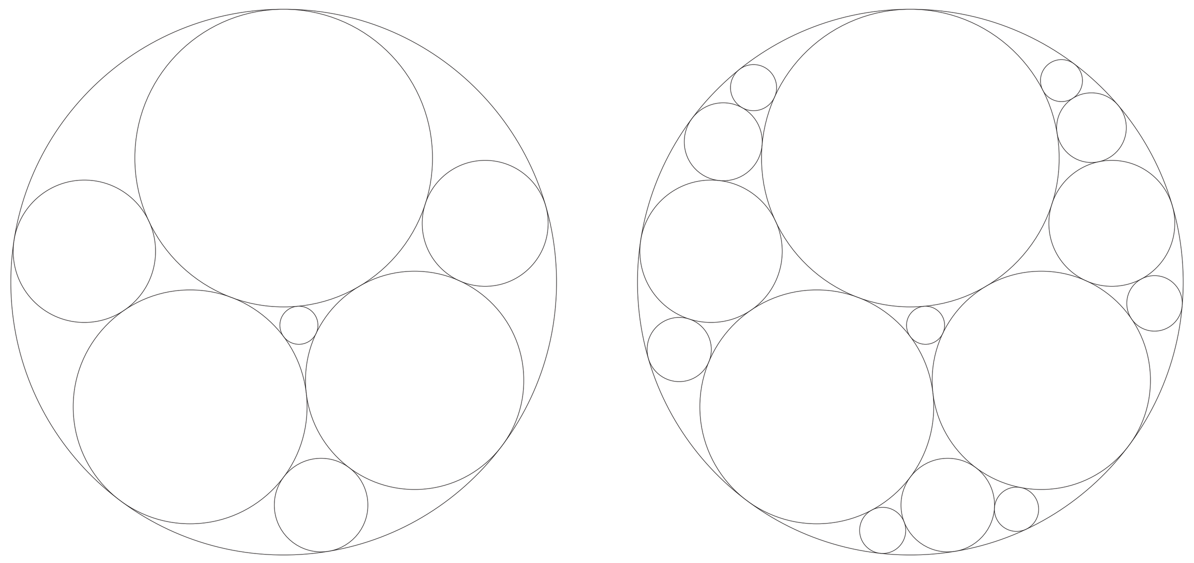


Descartes quadruple: a set of four mutually tangent circles with disjoint interiors.
Theorem of Apollonius: If three circles are mutually tangent, there are two circles that are tangent to all three.

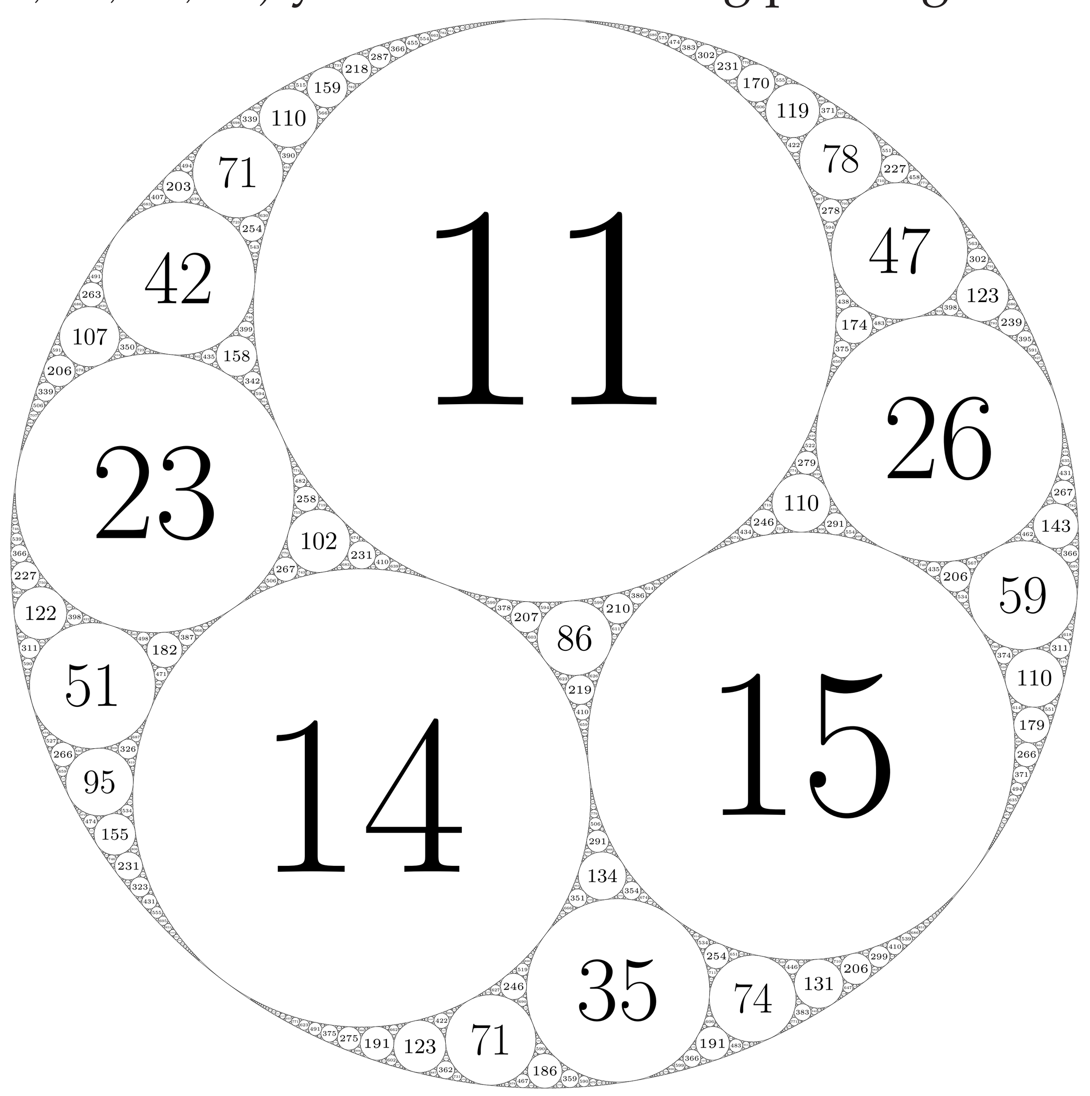
The **curvature** of a circle with radius r is defined to be $1/r$. **Descartes equation:** If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

Starting with three such circles and adding in the two circles of Apollonius, we obtain five circles. Repeating this process we can "fill" the circle, creating an Apollonian circle packing.



If $a, b, c, d \in \mathbb{Z}$, then the rest of the packing is also integral! If $\gcd(a, b, c, d) = 1$, the packing is **primitive**. For example, the Descartes quadruple $(-6, 11, 14, 15)$ yields the following packing:



THE LOCAL-GLOBAL CONJECTURE

Let $\mathcal{A}$ be a primitive Apollonian circle packing. The curvatures in $\mathcal{A}$ mod 24 fall into a set of six or eight possible residues, called **admissible residues**. Let $R(\mathcal{A})$ be the set of residues mod 24 of the curvatures in $\mathcal{A}$, one of six possible sets. The set $R(\mathcal{A})$ is called the **admissible set** of the packing, labelled by a **type** (x, k) where x is its size and k is the smallest residue in R that is coprime to 24.

The Local-Global Conjecture ([GLM+03, FS11]). The set of positive integers $R(\mathcal{A}) \pmod{24}$ not occurring in $\mathcal{A}$ is finite.

Type	$R(\mathcal{A})$
$(6,1)$	$0,1,4,9,12,16$
$(6,5)$	$0,5,8,12,20,21$
$(6,13)$	$0,4,12,13,16,21$
$(6,17)$	$0,8,9,12,17,20$
$(8,7)$	$3,6,7,10,15,18,19,22$
$(8,11)$	$2,3,6,11,14,15,18,23$

THEOREM (HAAG-KERTZER-RICKARDS-STANGE).

The Local-Global Conjecture is false.

THE NEW CONJECTURE

Define the set $S_{d,u} := \{un^d : n \in \mathbb{Z}\}$ where u and d are positive integers. The set $S_{d,u}$ forms a **reciprocity obstruction** if infinitely many elements of $S_{d,u}$ are admissible in $\mathcal{A}$ mod 24, yet none appear in $\mathcal{A}$.

In order to differentiate the quadratic and quartic obstructions, we expand on the definition of type. There exist well-defined functions which relate to the possible curvatures of circles tangent to the input circle $\mathcal{C}$ and are constant across a packing containing $\mathcal{C}$:

$$\chi_2 : \{\text{circles in a primitive Apollonian circle packing}\} \to \{\pm 1\}$$

$$\chi_4 : \{\text{circles in a primitive Apollonian circle packing of type } (6,1) \text{ or } (6,17)\} \to \{1, i, -1, -i\}$$

The value of χ_2 determines the quadratic obstructions, and similarly χ_4 determines the quartic obstructions. We compute χ_2 in terms of the quadratic residuosity of the curvatures tangent to a circle, and χ_4 in terms of a quartic residue symbol. The **extended type** of an Apollonian circle packing $\mathcal{A}$ is either the triple (x, k, χ_2) or (x, k, χ_2, χ_4), where $\mathcal{A}$ has type (x, k) and corresponding values of χ_2 (and χ_4, if relevant).

The New Conjecture. The type of $\mathcal{A}$ implies the existence of certain quadratic and quartic obstructions:

Type	n^2 Obstructions	n^4 Obstructions	L-G false	L-G open
$(6,1,1,-1)$		$n^4, 4n^4, 9n^4, 36n^4$	$0,1,4,9,12,16$	
$(6,1,-1)$	$n^2, 2n^2, 3n^2, 6n^2$		$0,1,4,9,12,16$	
$(6,5,1)$	$2n^2, 3n^2$		$0,8,12$	$5,20,21$
$(6,5,-1)$	$n^2, 6n^2$		$0,12$	$5,8,20,21$
$(6,13,1)$	$2n^2, 6n^2$		0	$4,12,13,16,21$
$(6,13,-1)$	$n^2, 3n^2$		$0,4,12,16$	$13,21$
$(6,17,1,1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0,9,12$	$8,17,20$
$(6,17,1,-1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0,9,12$	$8,17,20$
$(6,17,-1)$	$n^2, 2n^2$		$0,8,9,12$	$17,20$
$(8,7,1)$	$3n^2, 6n^2$		$3,6$	$7,10,15,18,19,22$
$(8,7,-1)$	$2n^2$		18	$3,6,7,10,15,19,22$
$(8,11,-1)$	$2n^2, 3n^2, 6n^2$		$2,3,6,18$	$11,14,15,23$

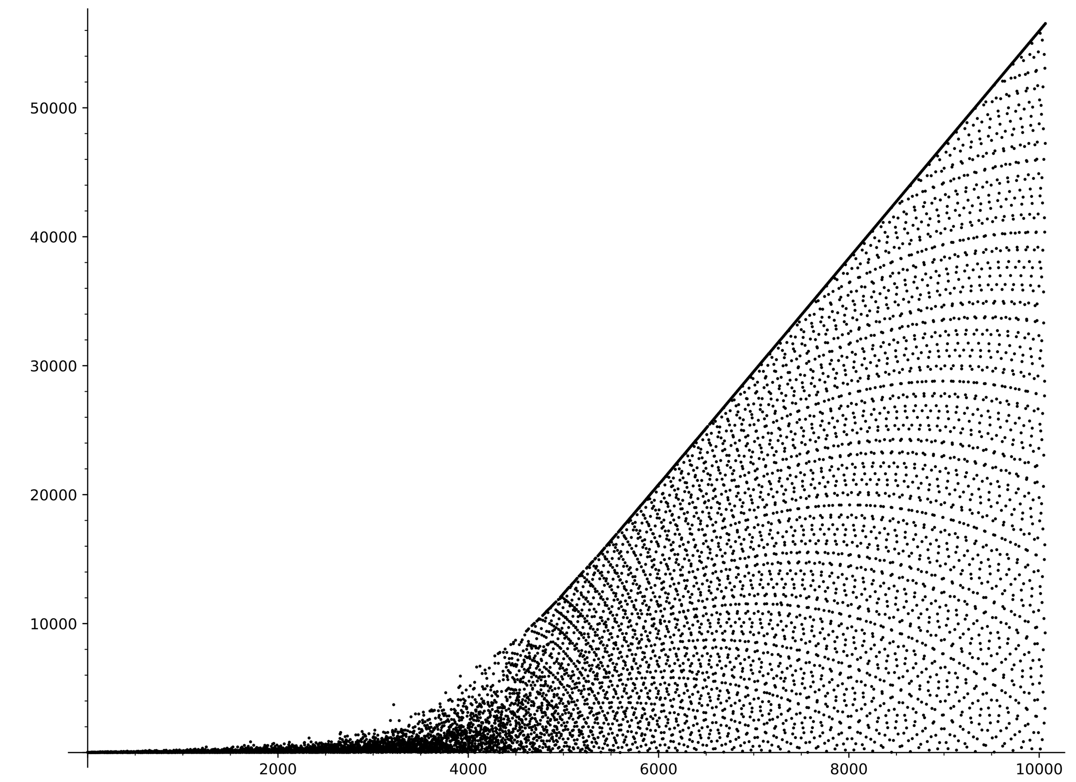
Types $(6,1,1,1)$ and $(8,11,1)$ have no obstructions and are L-G open for all admissible residues.

COMPUTATIONS

Define $S_{\mathcal{A}}$ to be the "sporadic set" of missing curvatures which are not in the quadratic or quartic obstructions, and $S_{\mathcal{A}}(N)$ to be the set of sporadic curvatures in $\mathcal{A}$ that are at most a positive integer N. We have computed missing curvatures $S_{\mathcal{A}}(N)$ for various packings $\mathcal{A}$ in the range $[10^{10}, 10^{12}]$ which strongly suggests that $S_{\mathcal{A}}$ is finite.

The following tables shows two examples of computation with $N = 10^{10}$:

Packing	$(-1,2,2,3)$	$(-3,5,8,8)$
Type	$(8,11,1)$	$(6,5,-1)$
Obstructions	none	$n^2, 6n^2$
$\lvert S_{\mathcal{A}}(N)\rvert$	61	676
$\max \lvert S_{\mathcal{A}}(N)\rvert$	$97,287$	$3,122,880$
$\approx \frac{N}{\max(S_{\mathcal{A}}(N))}$	$102,789$	$3,202$



Successive differences of missing curvatures in the packing $(-4, 5, 20, 21)$. The quadratic families $2n^2$ and $3n^2$ begin to predominate (the sporadic set has 3659 elements $< 10^{10}$, and occur increasingly sparsely.)

REFERENCES

[FS11] Elena Fuchs and Katherine Sanden. Some experiments with integral Apollonian circle packings. *Exp. Math.*, 20(4):380–399, 2011.

[GLM+03] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian Circle Packings: Number Theory. *J. Number Theory*, 100(1):1–45, 2003.