Symmetries Within Symmetries: Circle Packings

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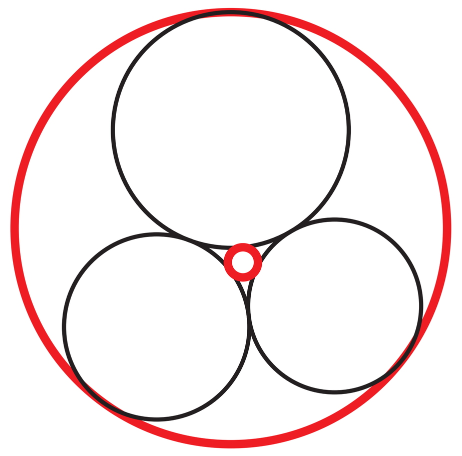
Apollonian Circle Packings

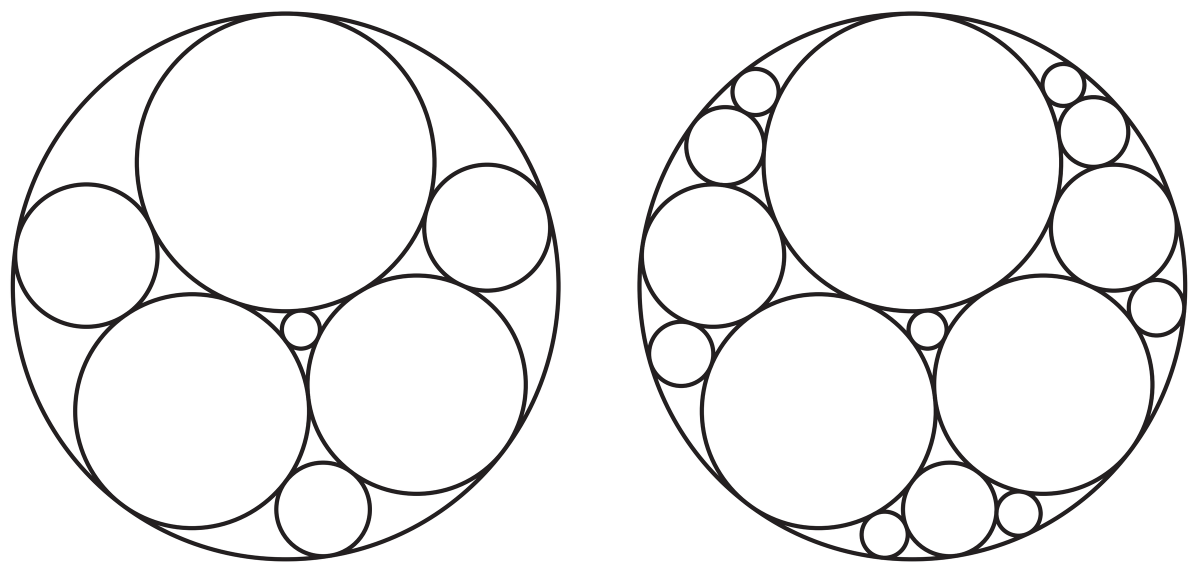
Descartes quadruple: a set of four mutually tangent circles with disjoint interiors.
Theorem of Apollonius: If three circles are mutually tangent, there are two circles that are tangent to all three.



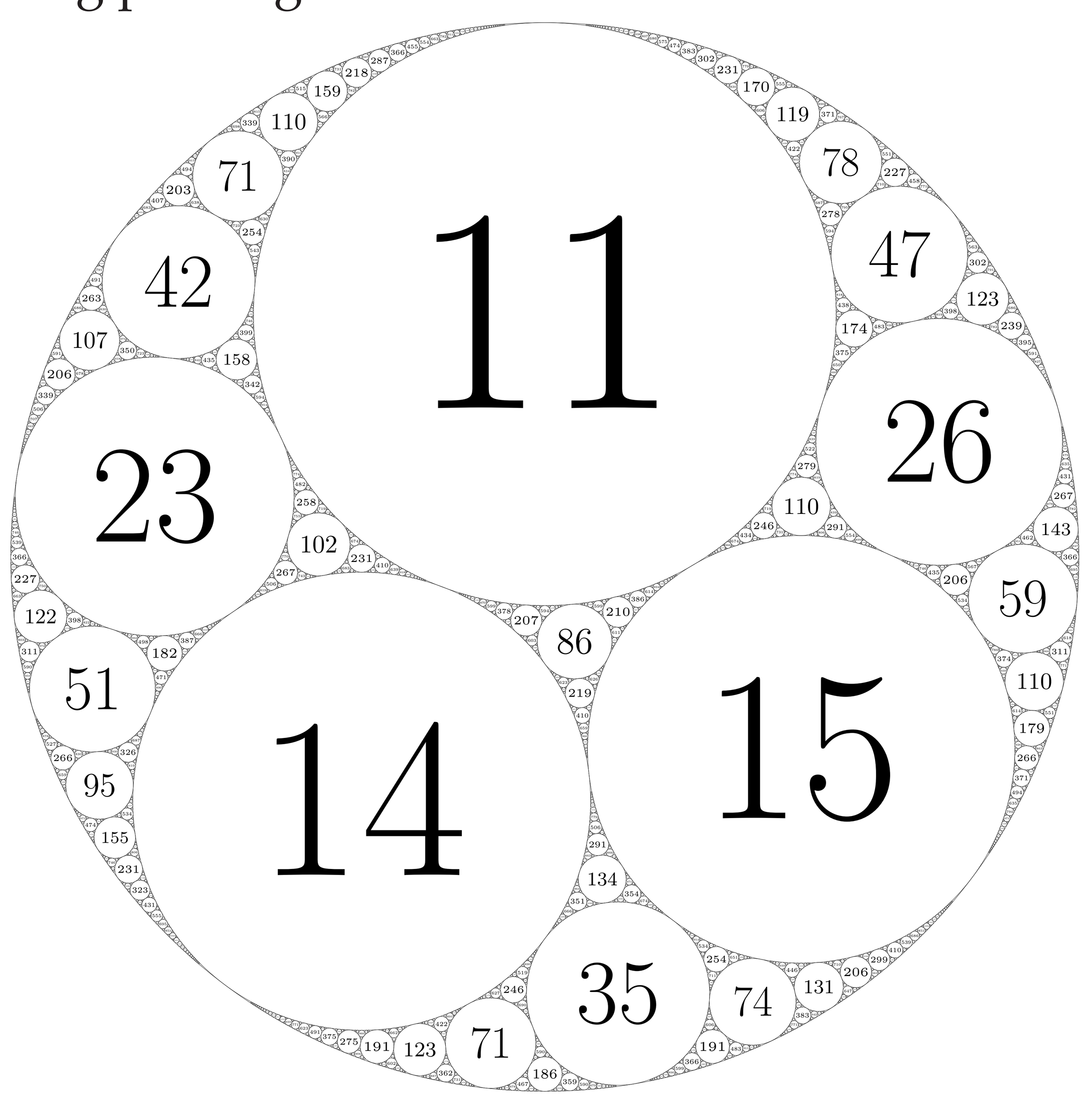
The **curvature** of a circle with radius r is defined to be $1/r$. **Descartes equation:** If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

Starting with three such circles and adding in the two circles of Apollonius, we obtain five circles. Repeating this process we can "fill" the circle, creating an Apollonian circle packing.

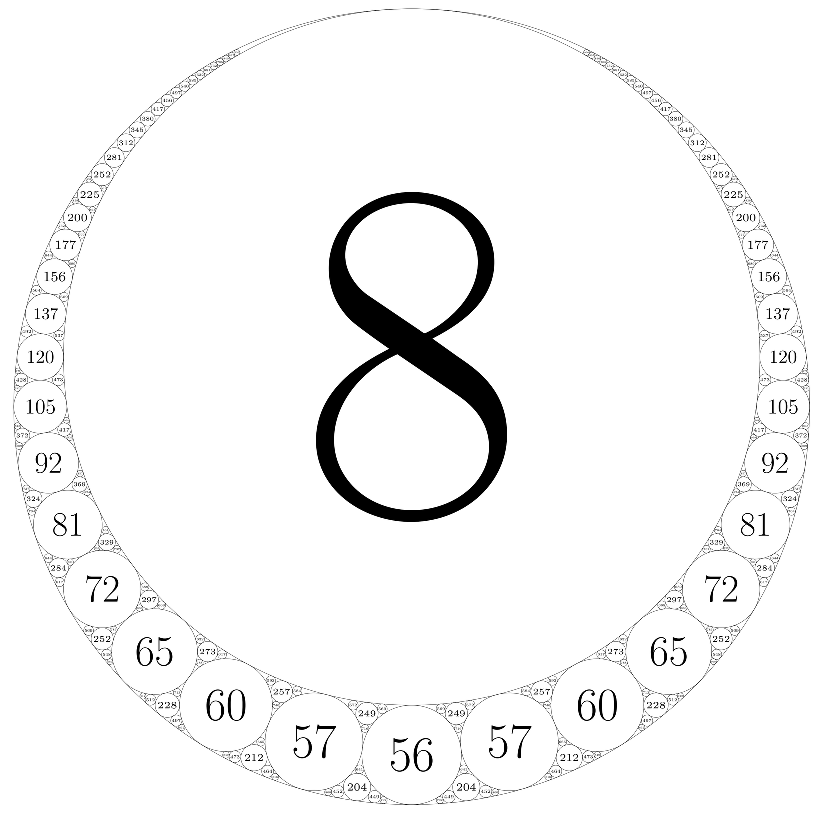


If the Descartes quadruple is integral, then the rest of the packing is also integral! For example, the Descartes quadruple $(-6, 11, 14, 15)$ yields the following packing:

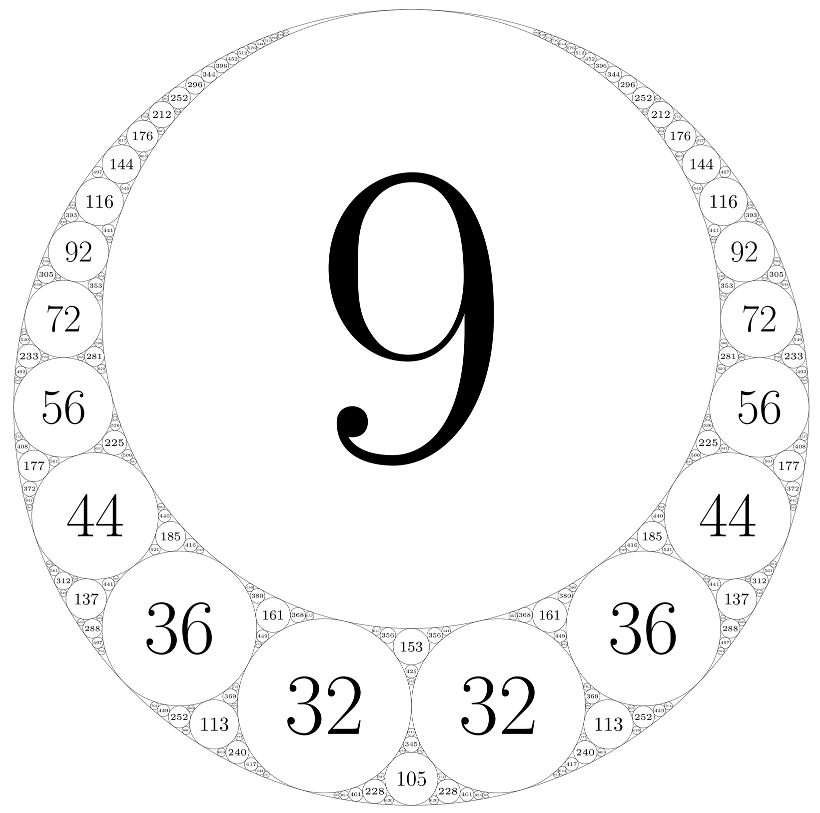


The three packings of -7

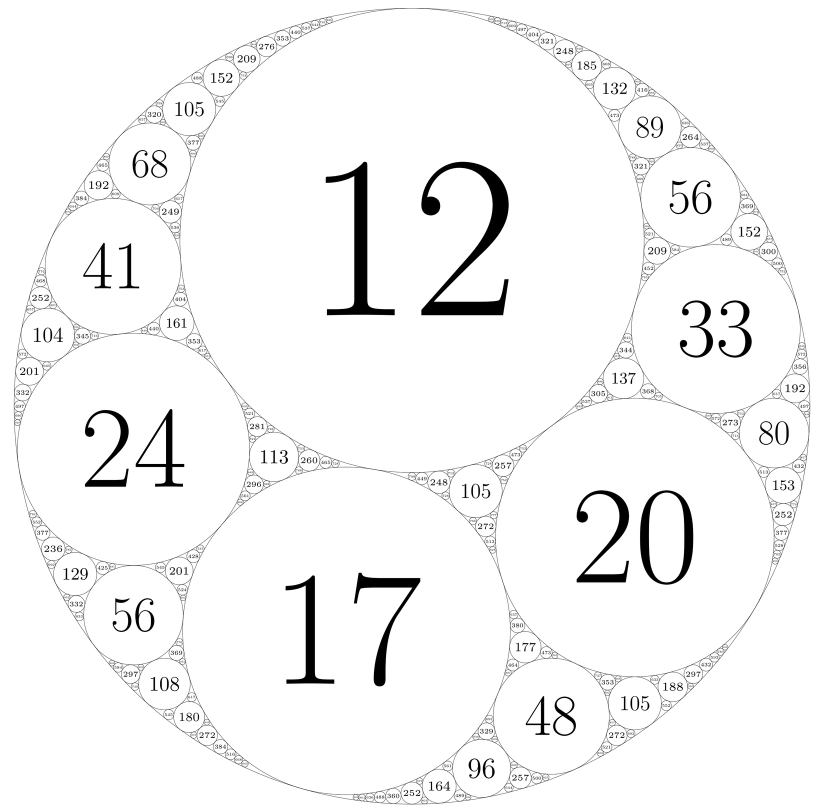
Take a circle of radius $1/7$. Restricting to radii with integer curvatures (the reciprocal of the radius), there are three ways that three mutually tangent circles can lie tangent inside the enclosing circle:



$[-7, 8, 56, 57]$



$[-7, 9, 32, 32]$

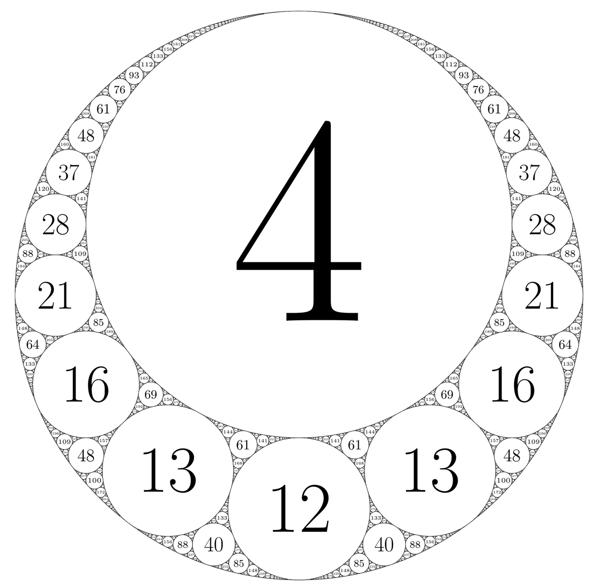


$[-7, 12, 17, 20]$

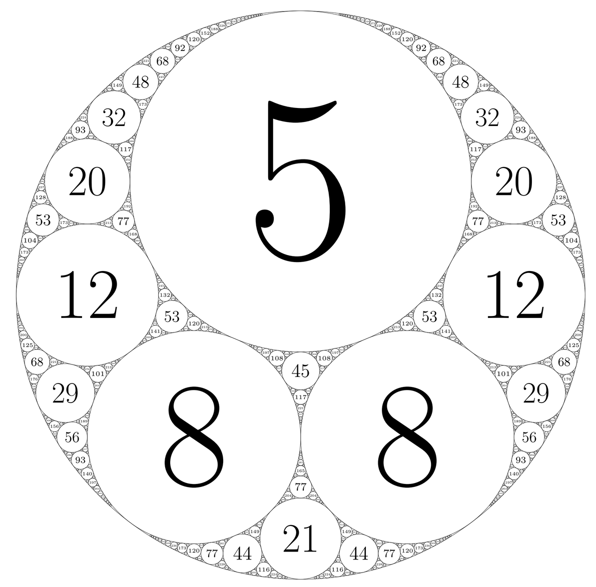
Symmetric Circle Packings

A packing is **sum-symmetric** if it has a line of symmetry that is not tangent to any circles. This gives $a + b + c = d$.

A packing is **twin-symmetric** if it has a line of symmetry tangent to two circles with equal curvature. This gives $c = d$ or $c = b$.



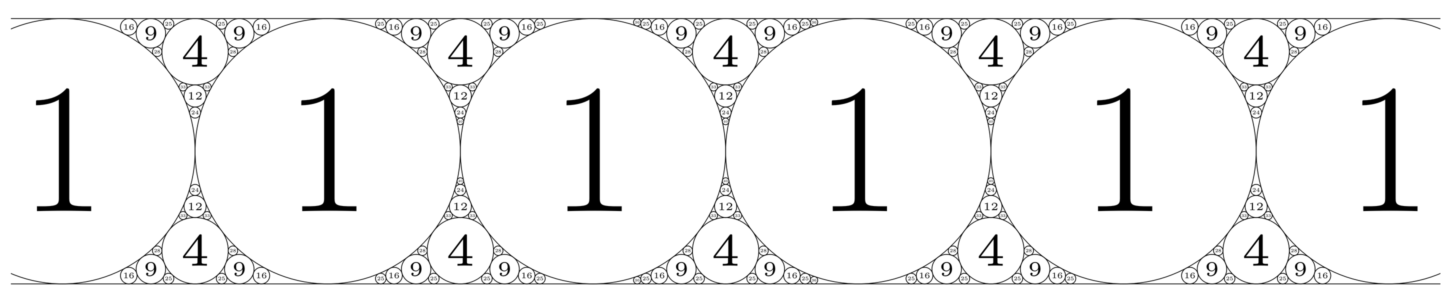
$[-3, 4, 12, 13]$



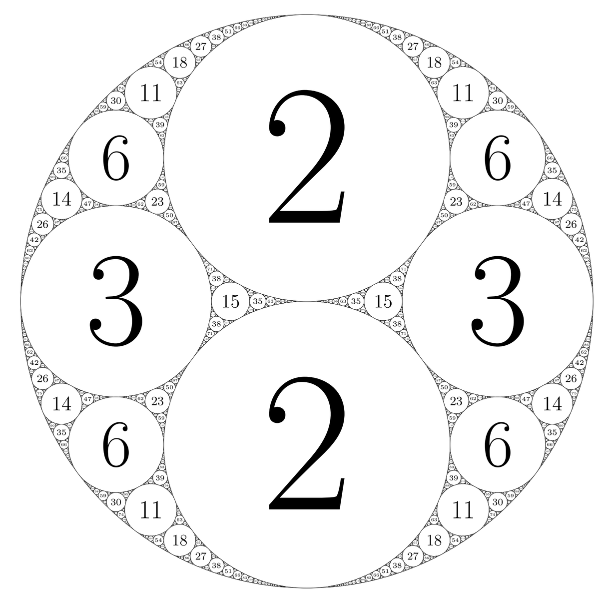
$[-3, 5, 8, 8]$

Proposition (Kertzer, 2024). *A symmetric packing is either sum-symmetric or twin-symmetric.*

There are two symmetric packings that are *both sum-symmetric and twin symmetric*: the strip and bug-eye packing.



The strip packing: $[0, 0, 1, 1]$



The bug-eye packing: $[-1, 2, 2, 3]$

Paramaterizations

Theorem (Kertzer, 2024). *A primitive **sum-symmetric** quadruple $[a, b, c, d]$ has the form*

$$[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy]$$

with $\gcd(x, y) = 1$, and $x, y \geqslant 0$.

Corollary. *There are $2^{\omega(n)-1}$ **sum-symmetric** packings for a given enclosing curvature of $-n$.*

$$\omega(n) = \#\{\text{distinct prime factors of } n\}$$

Example. Write $60 = 2^2 \cdot 3 \cdot 5$, with

$$2^{\omega(60)-1} = 2^{3-1} = 2^2 = 4$$

sum-symmetric packings. These correspond to the coprime factor pairs $(1, 60), (4, 15), (3, 20), (5, 12)$:

$$\begin{aligned}(1,60) &\implies [-60, 61, 3660, 3661]\\ (4,15) &\implies [-60, 76, 285, 301]\\ (3,20) &\implies [-60, 69, 460, 469]\\ (5,12) &\implies [-60, 85, 204, 229]\end{aligned}$$

Theorem (Kertzer, 2024). *A reduced **twin-symmetric** quadruple has one of following forms:*

$$\left[-xy,\ xy+2y^2,\ \frac{(x+y)^2}{2},\ \frac{(x+y)^2}{2}\right] \quad x, y \text{ odd}$$

$$[-2xy,\ 2xy+4y^2,\ (x+y)^2,\ (x+y)^2] \quad xy \text{ even}$$

with $\gcd(x, y) = 1$ and $x, y \geqslant 0$.

Corollary. *There are*

$$(1 - \delta_n) \cdot 2^{\omega(n)-1}$$

***twin-symmetric** packings for a given enclosing curvature of $-n$, where $\delta_n = 1$ if $n \equiv 2 \bmod 4$ and 0 otherwise*

References

Kertzer, Clyde. *Parameterizations of Descartes Quadruples*, CU Scholar, April 2024.