Apollonian Circle Packings

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June 15, 2023

Descartes Quadruples

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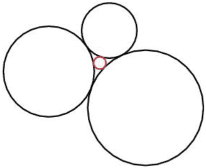

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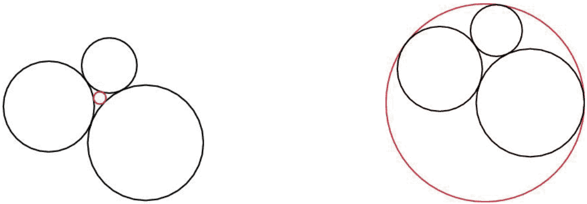

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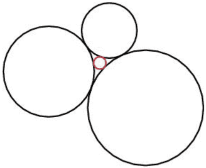

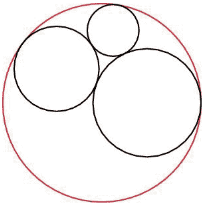

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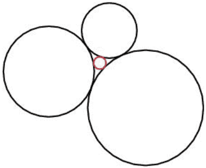

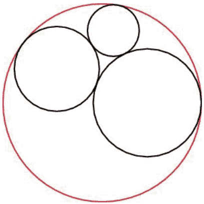

We can only have at most one "inverted" circle!

Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

The Descartes Equation

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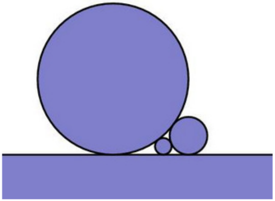

Quadruple with one circle of infinite radius

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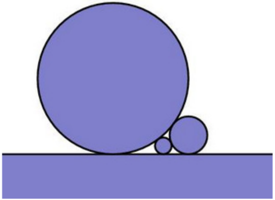

Quadruple with one circle of infinite radius

Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

Apollonian Circle Packings

If a, b, c, d are integers, the rest are also integers!

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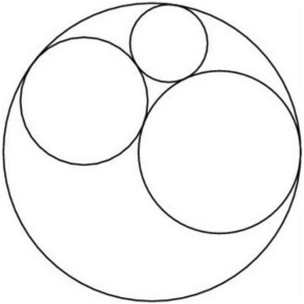

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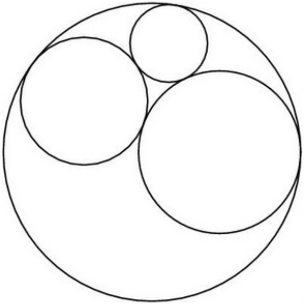

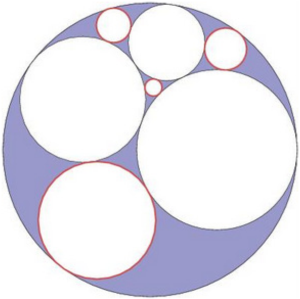

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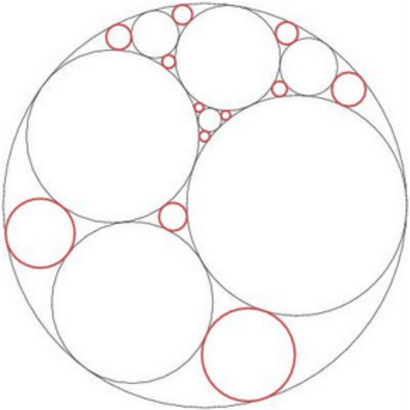

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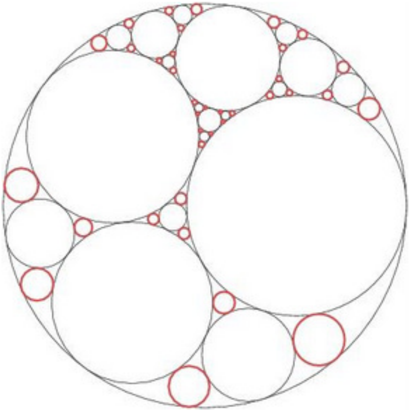

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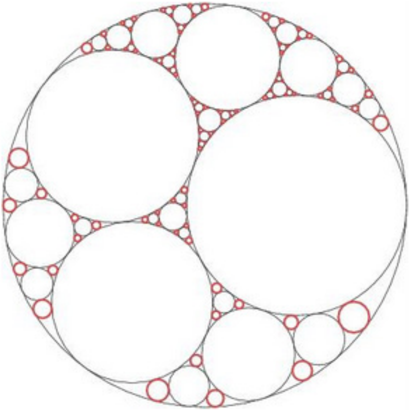

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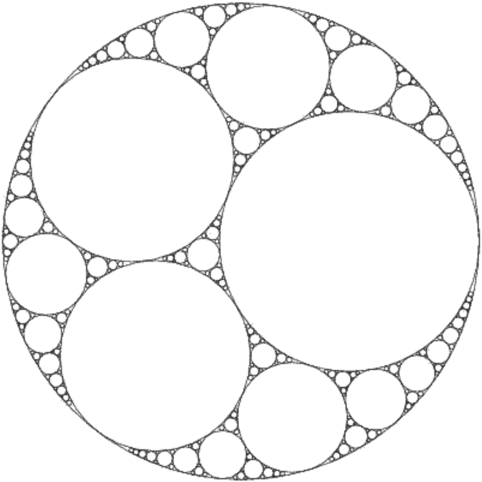

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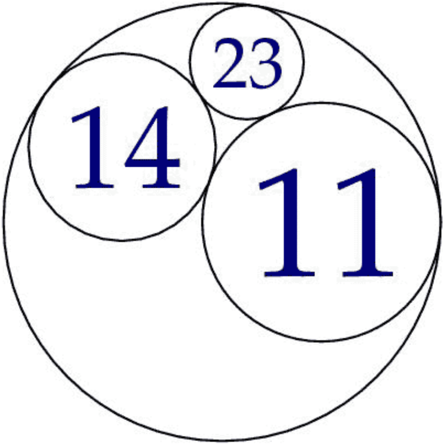
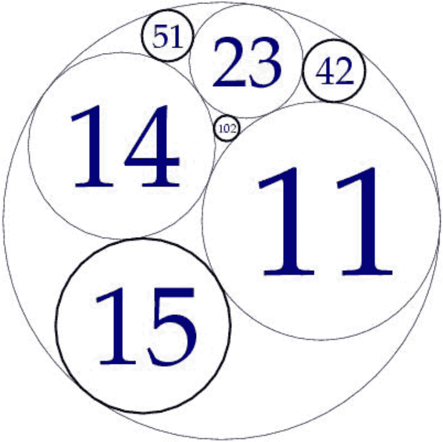


$[-6, 11, 14, 23]$[1]

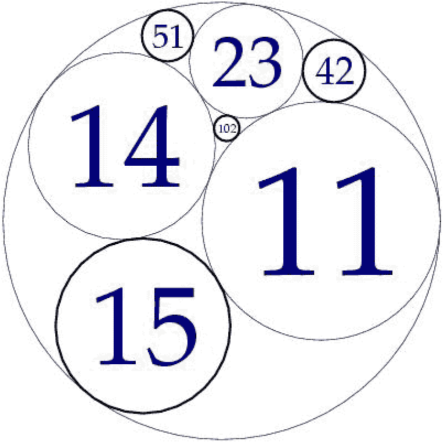
[1] Images from: AMS "When Kissing Involves Trigonometry"

Apollonian Circle Packings



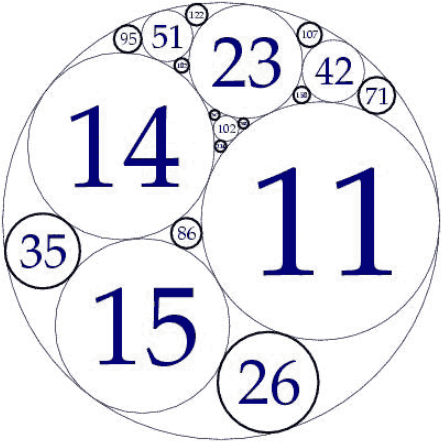
$[-6, 11, 14, 23]$

Apollonian Circle Packings



$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

Apollonian Circle Packings



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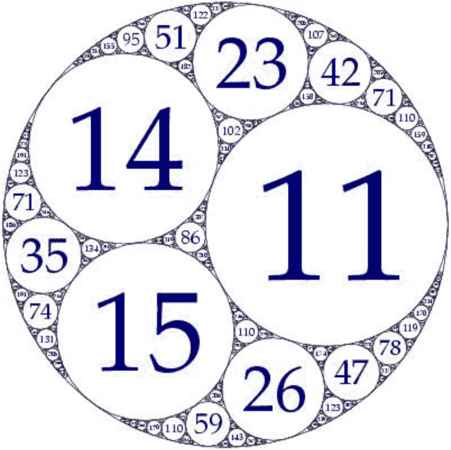

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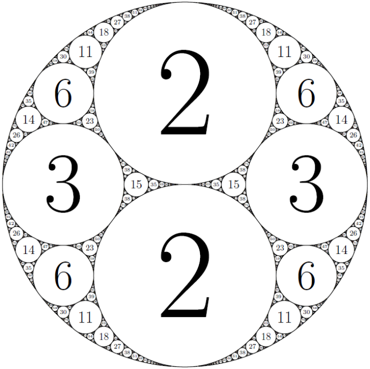
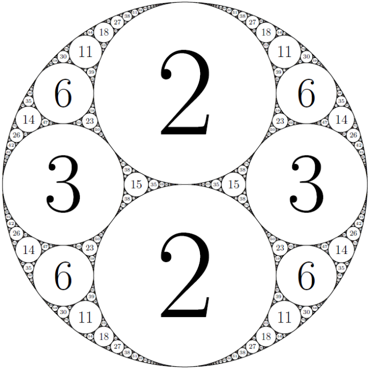


(a) $[-1, 2, 2, 3]$

Apollonian Circle Packings



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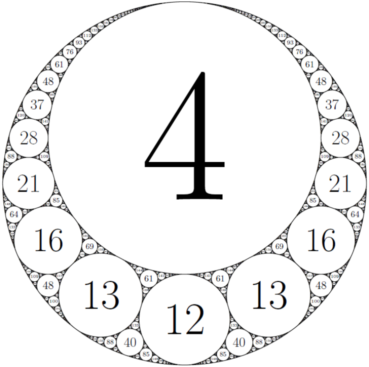
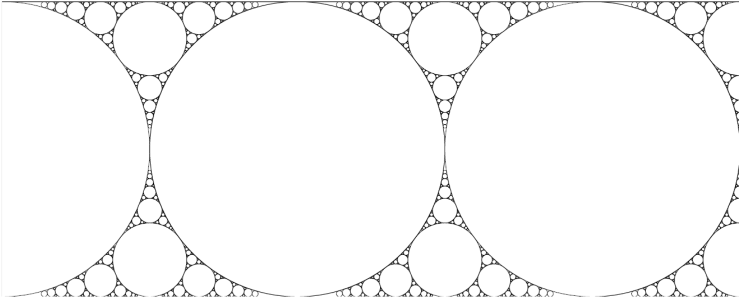


(b) $[-3, 4, 12, 13]$

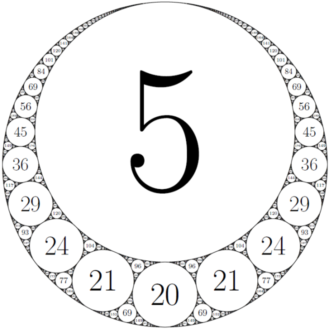
Apollonian Circle Packings



The strip packing: $[0, 0, 1, 1]$

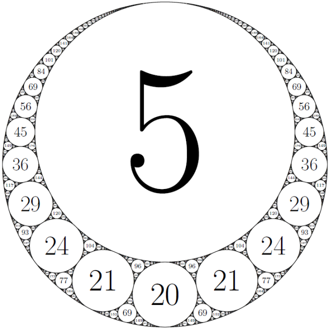
Symmetric Packings

Symmetric Packings



(a) $[-4, 5, 20, 21]$

Symmetric Packings



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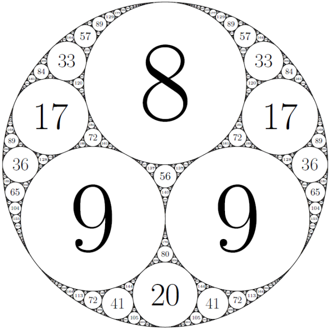

(b) $[-4, 8, 9, 9]$

Symmetric Packings

Symmetric Packings

Recall: $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$.

$$[-a, b, c, d] \quad d-c, \quad d-b, \quad d+a$$

Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
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$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81
$[-21, 30, 70, 79]$	9	49	100

Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
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Symmetric Packings

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Given the factorization of a, we can find the entire packing.

Symmetric Packings

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$$\left[-xy,\ x^2+xy,\ y^2+xy,\ (x+y)^2-xy\right]$$

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Try with $12 = 6 \cdot 2$:

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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

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$$\left[-xy,\ x(x+y),\ y(x+y),\ (x+y)^2 - xy\right] =$$

$$\left[-2 \cdot 6, 2(2+6), 6(2+6), (2+6)^2 - 2 \cdot 6\right] =$$

Symmetric Packings

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$[-12,\ 16,\ 48,\ 52]$

Symmetric Packings

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$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13]$$

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$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13] \quad (x = 3, y = 1)$$

Symmetric Packings

Theorem

All reduced primitive symmetric quadruples with distinct a, b, c, d are of the form

$$\left[-xy, x(x+y), y(x+y), (x+y)^2 - xy\right].$$

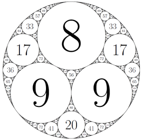
with $\gcd(x, y) = 1$.

Symmetric Packings

Packings where one of the numbers is the same:

Symmetric Packings

Packings where one of the numbers is the same: $[-4, 8, 9, 9]$



Symmetric Packings

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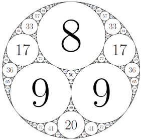

Theorem

All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

$$\left[-x,\ x+2y^2,\ 2\left(\frac{x+y^2}{2y}\right)^2,\ 2\left(\frac{x+y^2}{2y}\right)^2\right] \quad y \text{ odd}$$

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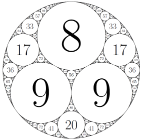

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