

Apollonian Circle Packings

Clyde Kertzer

University of Colorado Boulder

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Descartes Quadruples

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Descartes Quadruples

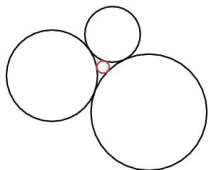
Definition

A *Descartes Quadruple* is a set of four mutually tangent circles with disjoint interiors.

Descartes Quadruples

Definition

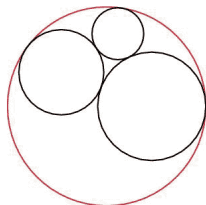
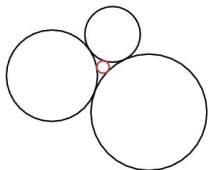
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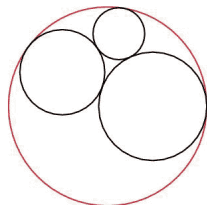
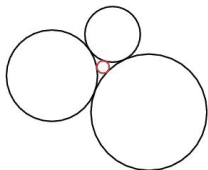
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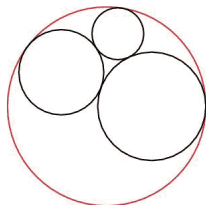
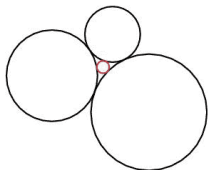


We can only have at most one "inverted" circle!

Descartes Quadruples

Definition

A *Descartes Quadruple* is a set of four mutually tangent circles with disjoint interiors.



We can only have at most one "inverted" circle!

Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

The Descartes Equation

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The Descartes Equation

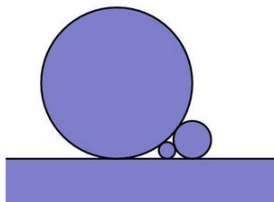
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The *curvature* of a circle with radius r is defined to be $1/r$.

The Descartes Equation

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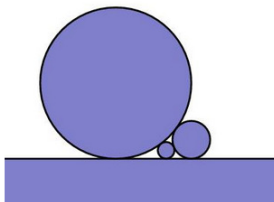


Quadruple with one circle of infinite radius

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Quadruple with one circle of infinite radius

Descartes Equation

If four mutually tangent circles have curvatures a , b , c , d then

$$(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2):$$

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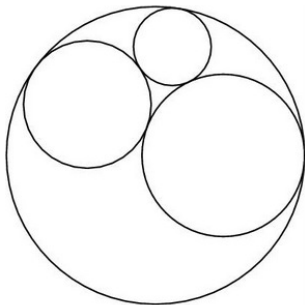
If $a; b; c; d$ are integers, the rest are also integers!

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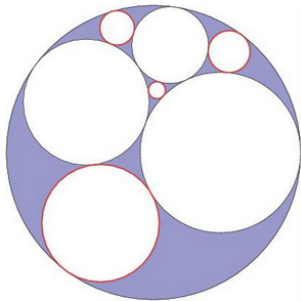
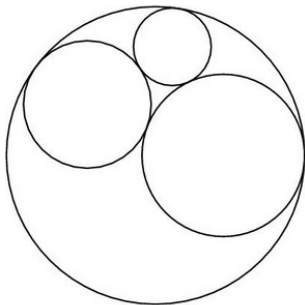


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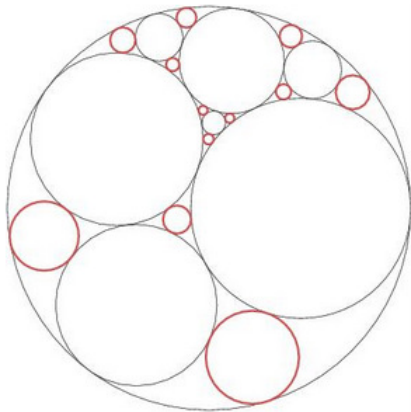
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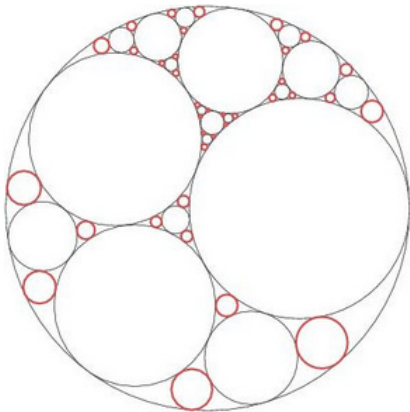
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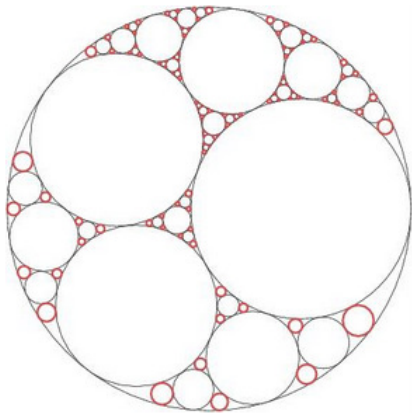
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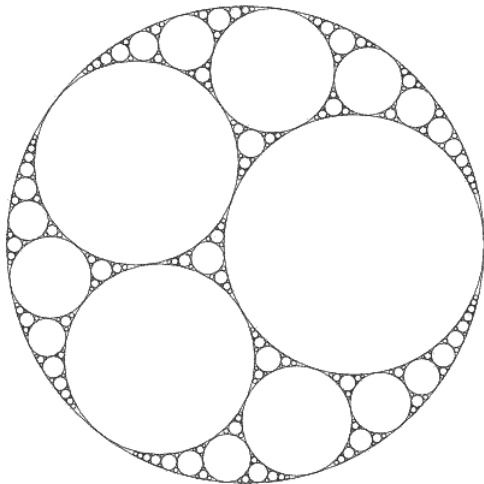
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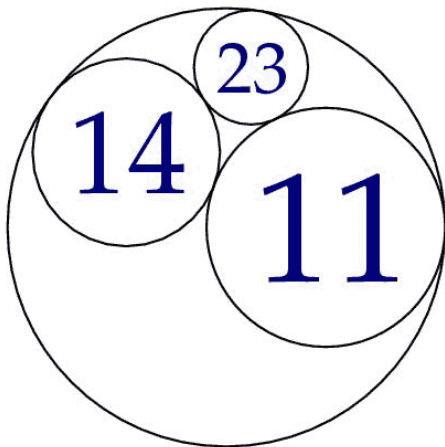
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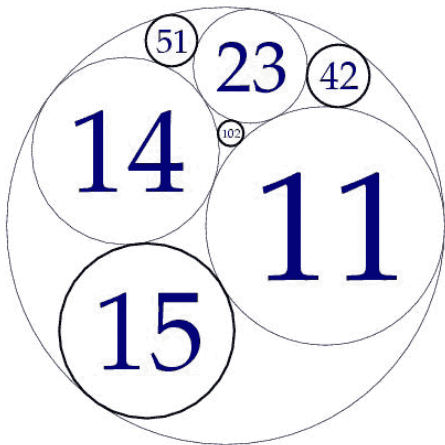
$$[6; 11; 14; 23]^1$$

¹Images from: AMS "When Kissing Involves Trigonometry"

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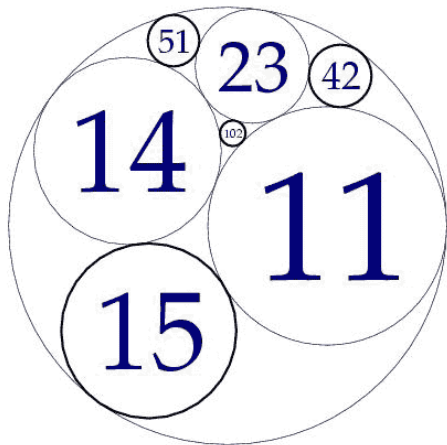


[6;11;14;23]

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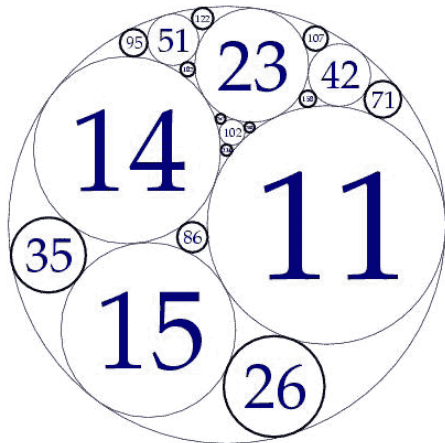


[6;11;14;23] reduces to [6;11;14;15]

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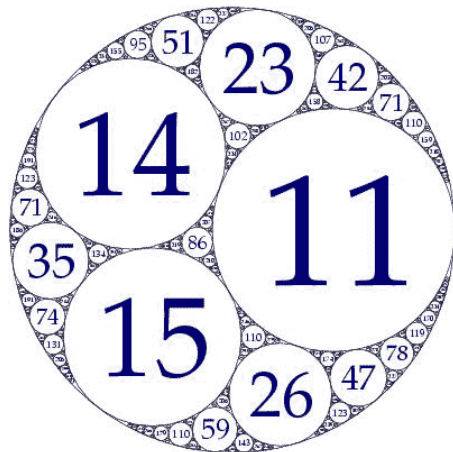


[6;11;14;15]

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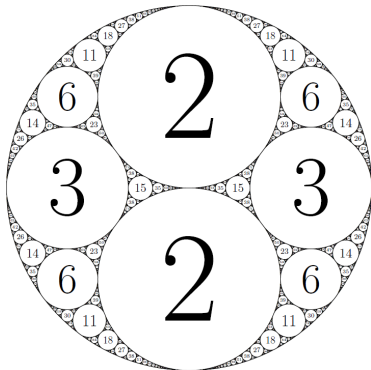


[6;11;14;15]

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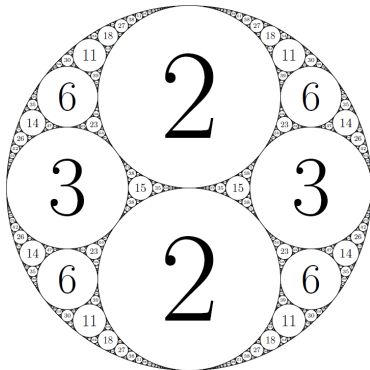


(a) [1;2;2;3]

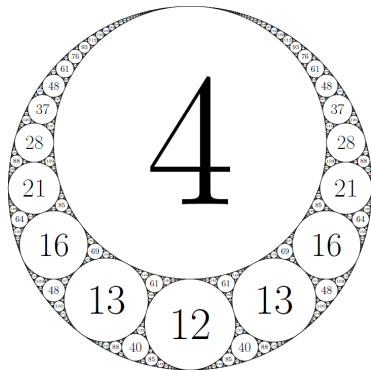
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(a) [1; 2; 2; 3]

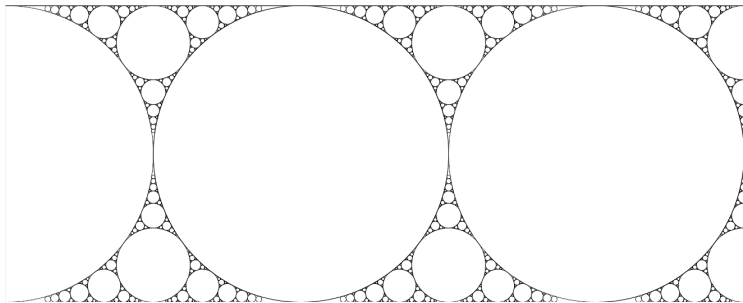


(b) [3; 4; 12; 13]

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The strip packing: $[0;0;1;1]$

Symmetric Packings

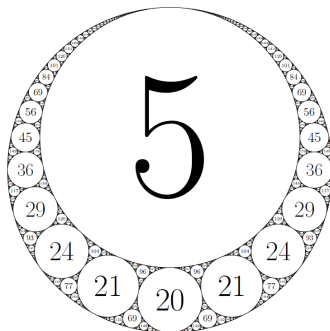
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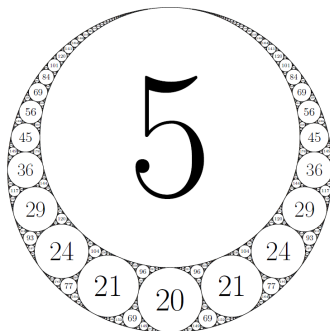


(a) [4;5;20;21]

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(a) [4;5;20;21]

(b) [4;8;9;9]

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Recall: $(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2)$:

$$[a; b; c; d] \quad d \quad c; \quad d \quad b; \quad d + a$$

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$$\begin{array}{c|ccc} [a; b; c; d] & d & c & d & b & d + a \\ \hline [6; 10; 15; 19] & & & & & \end{array}$$

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$$\begin{array}{c|c|c|c} [a; b; c; d] & d & c & d + a \\ \hline [6; 10; 15; 19] & 4 & 9 & 25 \end{array}$$

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$[a; b; c; d]$	d	c	d	b	$d + a$
$[6; 10; 15; 19]$	4		9		25
$[12; 21; 28; 37]$	9		16		49
$[18; 22; 99; 103]$	4		81		121
$[20; 36; 45; 61]$	16		25		81
$[21; 30; 70; 79]$	9		49		100

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$[a; b; c; d]$	$d \quad c$	$d \quad b$	$d + a$
$[6; 10; 15; 19]$	2^2	3^2	5^2
$[12; 21; 28; 37]$	3^2	4^2	7^2
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[$a; b; c; d$]	d c	b a	d b	c a	$d + a$
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Given the factorization of a , we can find the entire packing.

Symmetric Packings

$[a; b; c; d]$	$d \quad c$	$b \quad a$	$d \quad b$	$c \quad a$	$d + a$
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$$\begin{array}{c}
 2 \qquad \qquad \qquad 3 \\
 4 \quad \left| \frac{(2 \cdot 3)}{\underline{\{2\}}} \cdot \frac{2^2 + 2 \cdot 3}{\underline{\{2\}}} \cdot \frac{3^2 + 2 \cdot 3}{\underline{\{2\}}} \cdot \frac{(2+3)^2 + 2 \cdot 3}{\underline{\{2\}}} \right| 5 \\
 \qquad \qquad \qquad 6 \qquad \qquad 10 \qquad \qquad 15 \qquad \qquad 19
 \end{array}$$

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Given the factorization of a , we can find the entire packing.

$$\begin{array}{c}
 2 \qquad \qquad \qquad 3 \\
 4 \left(\frac{(2, 3)}{6}; \frac{2^2 + 2 \cdot 3}{10}; \frac{3^2 + 2 \cdot 3}{15}; \frac{(2+3)^2 - 2 \cdot 3}{19} \right)^5
 \end{array}$$

$$xy; x^2 + xy; y^2 + xy; (x + y)^2 \quad xy$$

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Given the factorization of a , we can find the entire packing.

2

3

$$4 \left| \frac{(2, 3)}{\underline{\{z\}}}; \frac{2^2 + 2, 3}{\underline{\{z\}}}; \frac{3^2 + 2, 3}{\underline{\{z\}}}; \frac{(2+3)^2 - 2, 3}{\underline{\{z\}}} \right| 3^5$$

6 10 15 19

$$xy; x^2 + xy; y^2 + xy; (x + y)^2 \quad xy$$

$$xy; x(x + y); y(x + y); (x + y)^2 \quad xy$$

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$[a; b; c; d]$	d	c	d	b	$d + a$
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Try with $12 = 6 \quad 2$:

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Try with $12 = 6 \cdot 2$:

$$xy; x(x+y); y(x+y); (x+y)^2 \quad xy =$$

$$2 \cdot 6; 2(2+6); 6(2+6); (2+6)^2 \quad 2 \cdot 6 =$$

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Try with $12 = 6 \quad 2$:

$$xy; x(x+y); y(x+y); (x+y)^2 \quad xy =$$

$$2 \quad 6; 2(2+6); 6(2+6); (2+6)^2 \quad 2 \quad 6 =$$

$$[12; 16; 48; 52]$$

Symmetric Packings

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$$2 \cdot 6; 2(2+6); 6(2+6); (2+6)^2 \quad 2 \cdot 6 =$$

$$[12; 16; 48; 52] = [3; 4; 12; 13]$$

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Try with $12 = 6 \cdot 2$:

$$xy; x(x+y); y(x+y); (x+y)^2 \quad xy =$$

$$2 \cdot 6; 2(2+6); 6(2+6); (2+6)^2 \quad 2 \cdot 6 =$$

$$[12; 16; 48; 52] = [3; 4; 12; 13] \quad (x = 3; y = 1)$$

Symmetric Packings

Theorem

All reduced primitive symmetric quadruples with distinct $a; b; c; d$ are of the form

$$xy; x(x+y); y(x+y); (x+y)^2 \quad xy :$$

with $\gcd(x; y) = 1$.

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Packings where one of the numbers is the same:

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Packings where one of the numbers is the same:

[4;8;9;9]

Symmetric Packings

Packings where one of the numbers is the same:
[4;8;9;9]

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Theorem

All primitive ACPs with $c = d$ are given by

$$\begin{aligned} & \text{"} \\ & x; x + y^2; \frac{2x + y^2}{2y}^2; \frac{2x + y^2}{2y}^{2\#} \quad y \text{ even} \\ & \text{"} \\ & x; x + 2y^2; 2 \frac{x + y^2}{2y}^2; 2 \frac{x + y^2}{2y}^{2\#} \quad y \text{ odd} \end{aligned}$$

Symmetric Packings

Packings where one of the numbers is the same:
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All primitive ACPs with $c = d$ are given by

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