Apollonian Circle Packings & Paramaterizations of Descartes Quaruples

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Descartes Quadruples

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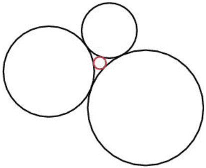

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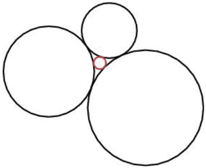

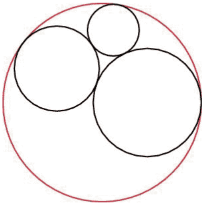

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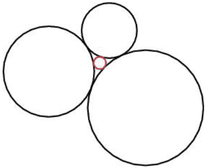

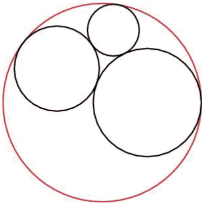

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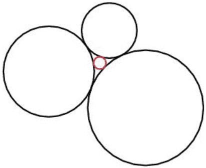

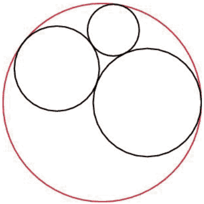

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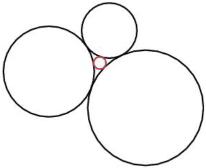

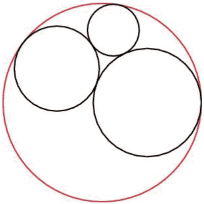

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Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

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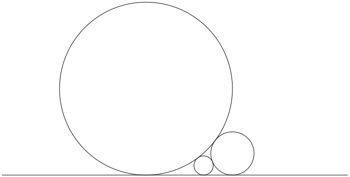

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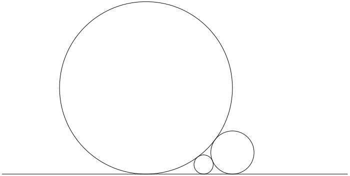

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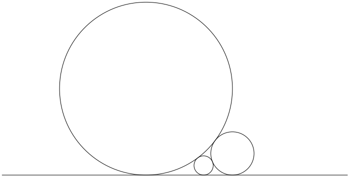

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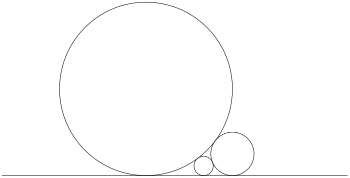

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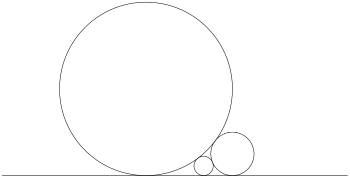

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$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

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$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \theta = 1 + 2 \cos \alpha \cos \beta \cos \theta.$$

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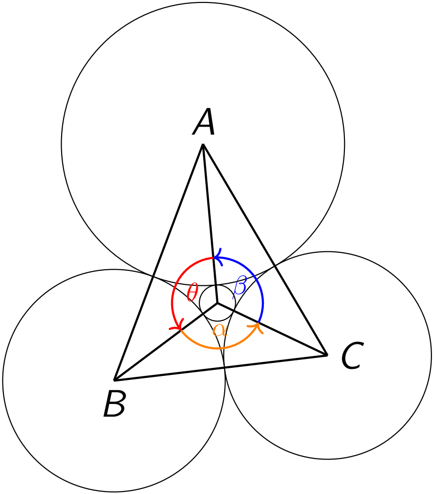
$$\begin{aligned}
&\cos^2\alpha + \cos^2\beta + \cos^2\theta = \\
&\qquad = \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\theta}{2} \\
&\qquad = \frac{3}{2} + \frac{\cos 2\alpha + \cos 2\beta}{2} + \frac{\cos(2\pi - (2\alpha + 2\beta))}{2} \\
&\qquad = \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{\cos 2(\alpha+\beta)}{2} \\
&\qquad = \frac{3}{2} + \cos(\alpha+\beta)\cos(\alpha-\beta) + \frac{2\cos^2(\alpha+\beta) - 1}{2} \\
&\qquad = 1 + \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos^2(\alpha+\beta) \\
&\qquad = 1 + (\cos(\alpha-\beta) + \cos(\alpha+\beta))\cos(2\pi - \theta) \\
&\qquad = 1 + 2\cos\alpha\cos\beta\cos\theta.
\end{aligned}$$

□

Proof of the Lemma



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$$AB = r_A + r_B, \quad BC = r_B + r_C, \quad AC = r_A + r_C$$

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Let $\angle BDC = \alpha$, $\angle CDA = \beta$, and $\angle ADB = \theta$. The law of cosines in $\triangle ADB$ yields

$$\begin{aligned}\cos\theta &= \frac{AD^2 + BD^2 - AB^2}{2 \cdot AD \cdot BD} \\ &= \frac{(r_A + r_D)^2 + (r_B + r_D)^2 - (r_A + r_B)^2}{2(r_A + r_D)(r_B + r_D)} \\ &= \frac{2r_D^2 + 2r_D(r_A + r_B) - 2r_A r_B}{2(r_A + r_D)(r_B + r_D)}\end{aligned}$$

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$$\cos\alpha = 1 - \frac{2r_B r_C}{(r_B + r_D)(r_C + r_D)}, \quad \cos\beta = 1 - \frac{2r_A r_C}{(r_A + r_D)(r_C + r_D)}.$$

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$$\cos\alpha = 1 - \frac{2k_D^2}{(k_B + k_D)(k_C + k_D)} = 1 - \lambda_\alpha$$

$$\cos\beta = 1 - \frac{2k_D^2}{(k_A + k_D)(k_C + k_D)} = 1 - \lambda_\beta$$

$$\cos\theta = 1 - \frac{2k_D^2}{(k_A + k_D)(k_B + k_D)} = 1 - \lambda_\theta.$$

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$$(1-\lambda_\alpha)^2 + (1-\lambda_\beta)^2 + (1-\lambda_\theta)^2 = 1 + 2(1-\lambda_\alpha)(1-\lambda_\beta)(1-\lambda_\theta)$$

$$\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\theta^2 + 2\lambda_\alpha\lambda_\beta\lambda_\theta = 2(\lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\theta + \lambda_\alpha\lambda_\theta)$$

$$\frac{\lambda_\alpha}{\lambda_\beta\lambda_\theta} + \frac{\lambda_\beta}{\lambda_\alpha\lambda_\theta} + \frac{\lambda_\theta}{\lambda_\alpha\lambda_\beta} + 2 = 2\left(\frac{1}{\lambda_\alpha} + \frac{1}{\lambda_\beta} + \frac{1}{\lambda_\theta}\right).$$

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Substituting back our values for the λs we find

$$\frac{(k_A+k_D)^2}{2k_D^2} + \frac{(k_B+k_D)^2}{2k_D^2} + \frac{(k_C+k_D)^2}{2k_D^2} + 2 =$$
$$2\frac{(k_B+k_D)(k_C+k_D)}{2k_D^2} + 2\frac{(k_A+k_D)(k_B+k_D)}{2k_D^2}$$
$$+\,2\frac{(k_A+k_D)(k_B+k_D)}{2k_D^2}.$$

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$$\begin{aligned} k_A^2 + k_B^2 + k_C^2 + 2k_D(k_A + k_B + k_C) + 7k_D^2 \\ = 6k_D^2 + 4k_D(k_A + k_B + k_C) \\ + 2(k_A k_B + k_B k_C + k_A k_C) \end{aligned}$$

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$$\begin{aligned} k_A^2 + k_B^2 + k_C^2 + k_D^2 &= 2k_D(k_A + k_B + k_C) \\ &\quad + 2(k_A k_B + k_B k_C + k_A k_C) \\ &= (k_A + k_B + k_C + k_D)^2 \\ &\quad - (k_A^2 + k_B^2 + k_C^2 + k_D^2) \\ 2(k_A^2 + k_B^2 + k_C^2 + k_D^2) &= (k_A + k_B + k_C + k_D)^2. \end{aligned}$$

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Moreover, $d + d' = 2(a + b + c)$.

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$$2\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2=0$$
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$$\begin{aligned} d &= (a+b+c) \\ &\quad \pm \frac{\sqrt{4(a+b+c)^2-4\left(a^2+b^2+c^2-2ab-2bc-2ac\right)}}{2} \\ &= a+b+c \pm 2\sqrt{ab+bc+ca}. \end{aligned}$$

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Thus, there are two options for d. Their sum is $2(a+b+c)$. □

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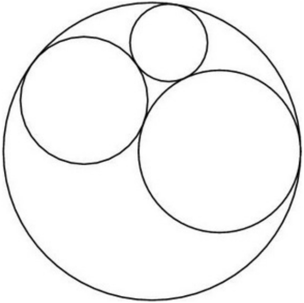

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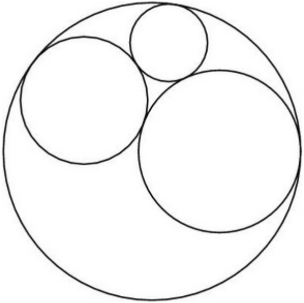

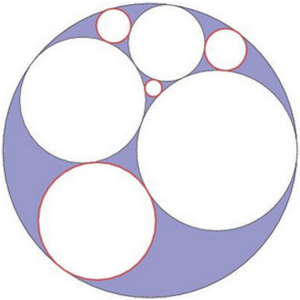

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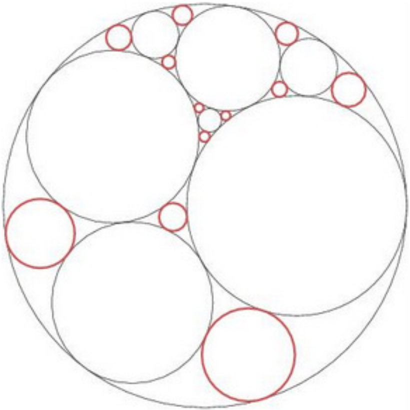

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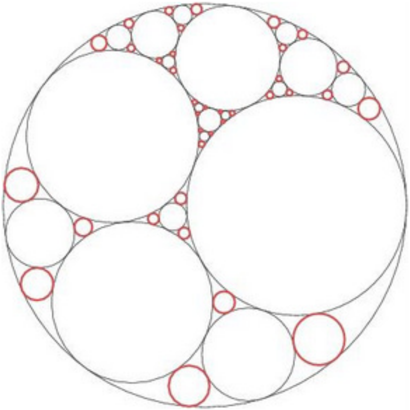

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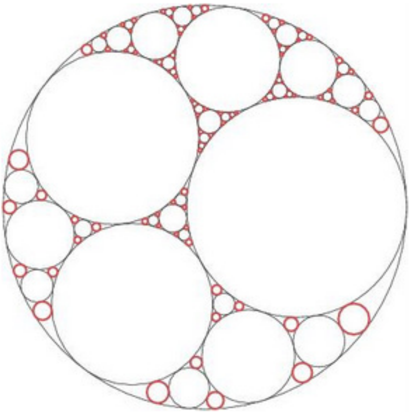

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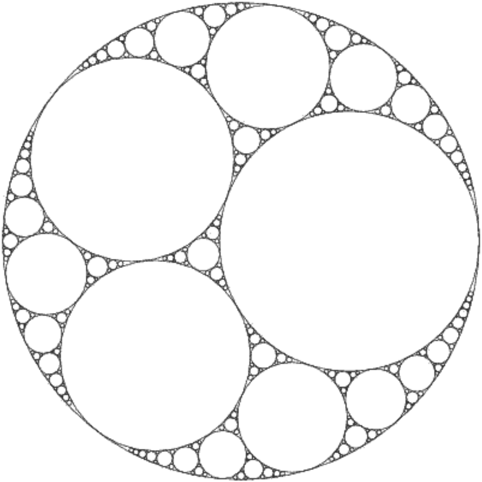

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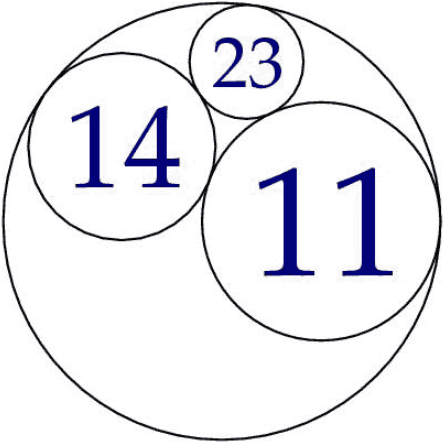

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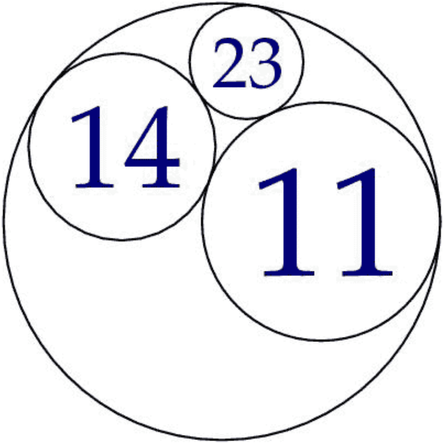

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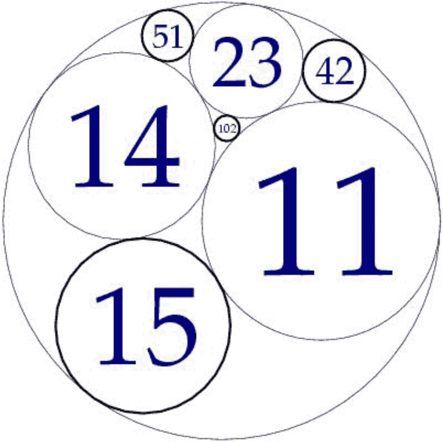

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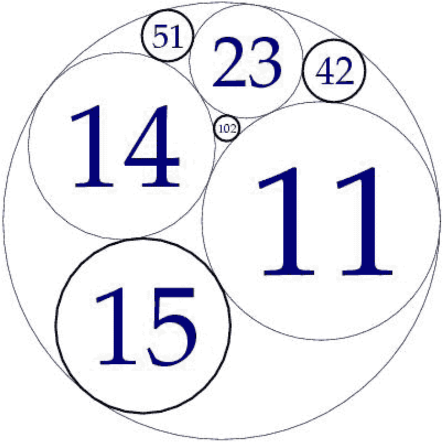
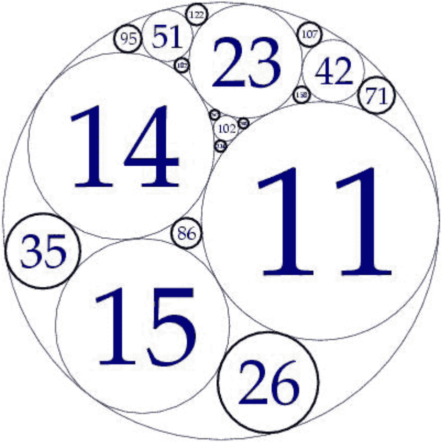


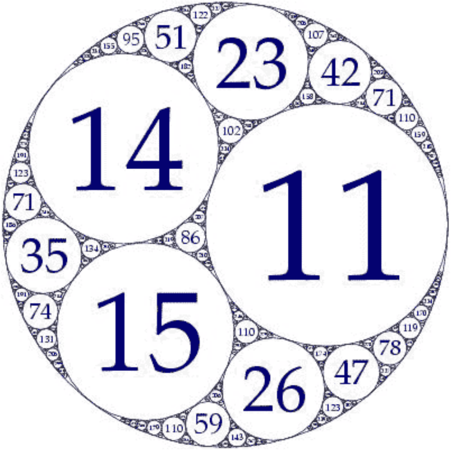
$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

Apollonian Circle Packings



$[-6, 11, 14, 15]$

Apollonian Circle Packings



$[-6, 11, 14, 15]$

Symmetric Packings

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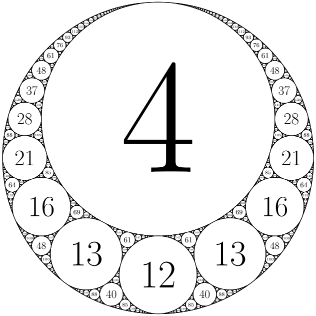
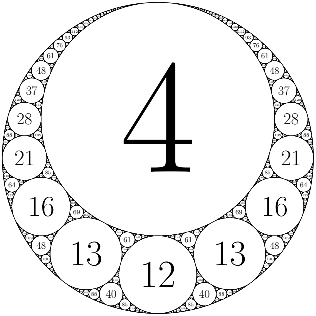
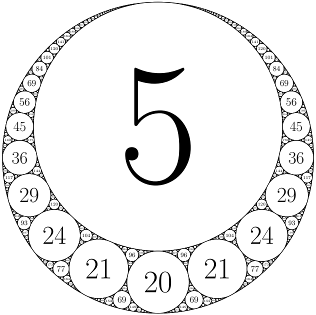


$[-3, 4, 12, 13]$

Symmetric Packings



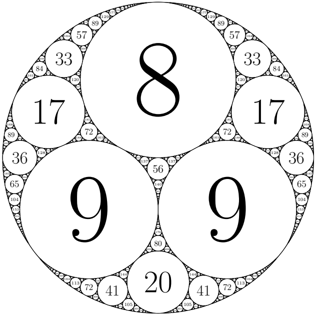
[−3, 4, 12, 13]



[−4, 5, 20, 21]

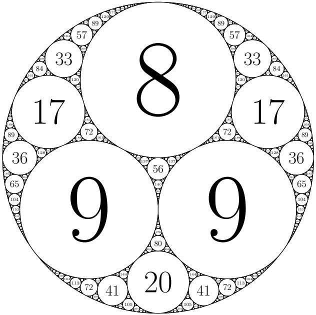
Symmetric Packings

Symmetric Packings

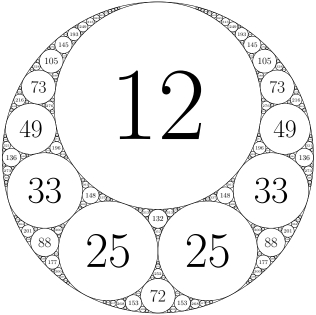


$[-4, 8, 9, 9]$

Symmetric Packings



[−4, 8, 9, 9]



[−8, 12, 25, 25]

Symmetric Packings Definitions

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A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$ and $a \leqslant 0 \leqslant b < c < d$. These packings have a line of symmetry that is not tangent to any circles.

$$\begin{aligned} 2(a + b + c) - d &= d \\ 2(a + b + c) &= 2d \\ a + b + c &= d \end{aligned}$$

The Two Unusual Symmetric Packings

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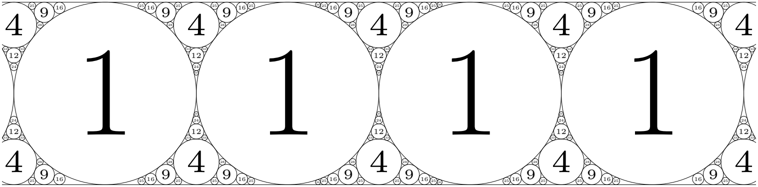

The strip packing: $[0, 0, 1, 1]$

The Two Unusual Symmetric Packings

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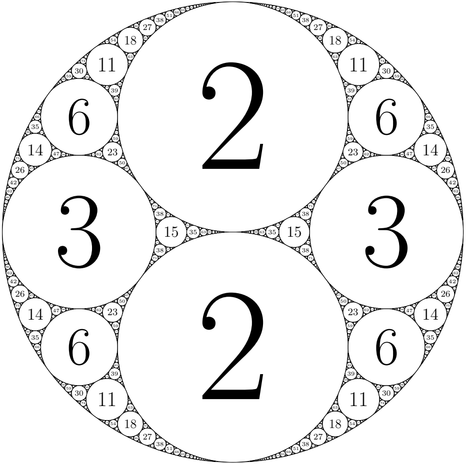

The bug-eye packing: $[-1, 2, 2, 3]$

Sum-Symmetric Packings

Let's recall the Descartes Equation:

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$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81
$[-21, 30, 70, 79]$	9	49	100

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
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Given the factorization of a, we can find the entire quadruple!

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$$[-12,\ 16,\ 48,\ 52]$$

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$$[-12,\ 16,\ 48,\ 52] = [-3,\ 4,\ 12,\ 13] \quad (x = 3, y = 1)$$

Sum-Symmetric Packing Paramaterization

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Sum-Symmetric Packing Paramaterization

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All reduced primitive symmetric quadruples with distinct a, b, c, d are of the form

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with $\gcd(x, y) = 1$.

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$$\begin{aligned} a + b + c &= d \\ a + b &= d - c. \end{aligned}$$

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(iii) Use substitutions from parts (i) and (ii) to find

$$\begin{aligned} a+b+c &= d \\ (a+b+c)^2 &= a^2+b^2+c^2 \\ a^2+b^2+c^2+2ab+2ac+2bc &= a^2+b^2+c^2 \\ ab+ac+bc &= 0 \end{aligned}$$

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Let $g = \gcd(a + b, a + c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y. This yields $gxy = -a$.

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Suppose that $[a, b, c, d]$ is a reduced primitive symmetric quadruple such that $a < 0 < b < c < d$. Adding a^2 to both sides of Proposition part (iii) we have

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$$a^2 + ab + ac + bc = a^2$$
$$(a + b)(a + c) = a^2.$$

Let $g = \gcd(a + b, a + c)$ so that $a + b = gx^2$ and $a + c = gy^2$ for some integers x and y. This yields $gxy = -a$. Now, we have

$$b = (a + b) + (-a) = gx^2 + gxy$$

Proving the Paramaterization

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and

$$c = (a + c) + (-a) = gy^2 + gxy.$$

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Thus, we have that

$$\begin{aligned} a &= -gxy \\ b &= gx(x+y) \\ c &= gy(x+y) \\ d &= g((x+y)^2 - xy). \end{aligned}$$

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$$\begin{aligned} a &= -xy \\ b &= x(x+y) \\ c &= y(x+y) \\ d &= (x+y)^2 - xy. \end{aligned}$$

with $\gcd(x, y) = 1$. □

The Number of Sum-Symmetric Packings

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We say a positive integer *a has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

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$$[-7, 8, 56, 57], \quad [-7, 12, 17, 20], \quad [-7, 9, 32, 32].$$

The Number of Sum-Symmetric Packings

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Sum-Symmetric packings of 60

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These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$. They are

$$\begin{aligned}(1, 60) &\implies [-60, 61, 3660, 3661]\\(4, 15) &\implies [-60, 76, 285, 301]\\(3, 20) &\implies [-60, 69, 460, 469]\\(5, 12) &\implies [-60, 85, 204, 229]\end{aligned}$$

Twin-Symmetric Packings

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Packings where one of the numbers is the same:

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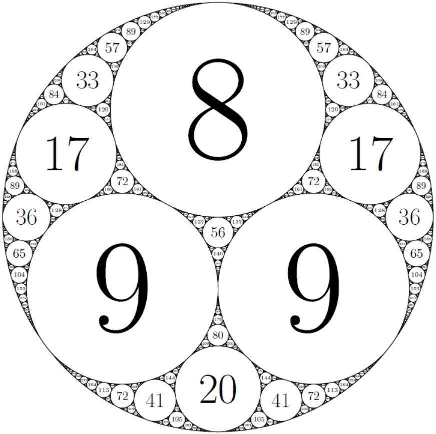

$[-4, 8, 9, 9]$

Twin-Symmetric Packings

Twin-Symmetric Packings

-2	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
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-9	$[-9, 11, 50, 50]$
-10	none

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-2	none
-3	$[-3, 5, 8, 8]$
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-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$

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-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
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-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
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-10	none
-11	$[-11, 13, 72, 72]$
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-13	$[-13, 15, 98, 98]$
-14	none

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-2	none
-3	$[-3, 5, 8, 8]$
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-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
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-14	none
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-16	$[-16, 20, 81, 81]$

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Over the summer:

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All primitive ACPs with $c = d$ are given by

$$\left[-x,\ x+y^2,\ \left(\frac{2x+y^2}{2y}\right)^2,\ \left(\frac{2x+y^2}{2y}\right)^2\right] \quad y \text{ even}$$

$$\left[-x,\ x+2y^2,\ 2\left(\frac{x+y^2}{2y}\right)^2,\ 2\left(\frac{x+y^2}{2y}\right)^2\right] \quad y \text{ odd}$$

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Not ideal, not in terms of factorization.

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Improved to:

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A twin-symmetric quadruple is of the form

$$\begin{cases} \left[-xy,\ xy+2y^2,\ \frac{(x+y)^2}{2},\ \frac{(x+y)^2}{2}\right] & x \text{ odd},\ y \text{ odd} \quad x > y \\ \\ \\ \end{cases}$$

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with $\gcd(x, y) = 1$.

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Ex: $x=4,\ y=3$

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Why won't $x=2, y=3$ work?

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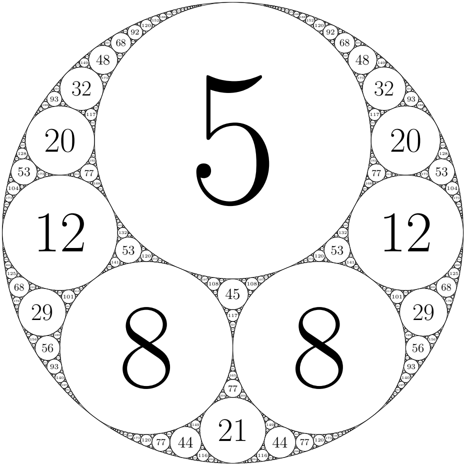
$$[-6, 6+4(3)^2, 4^2, 4^2] \implies [-6, 42, 16, 16] \implies [-3, 21, 8, 8]$$

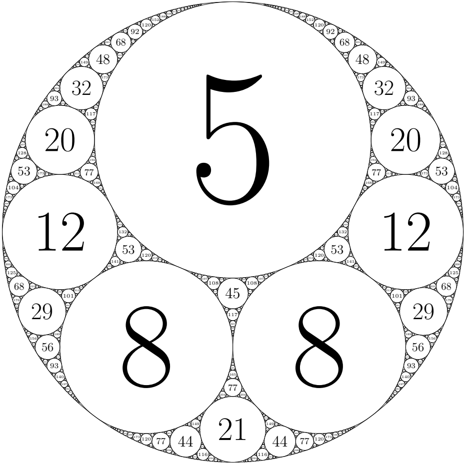
Twin-symmetric Packings

Twin-symmetric Packings



$[-3, 21, 8, 8]$

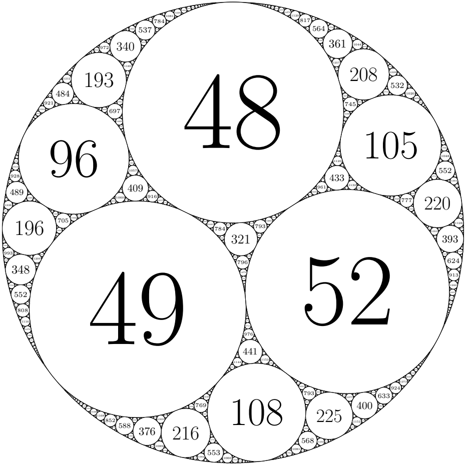
Twin-symmetric Packings



$$[-3, 21, 8, 8] \implies [-3, 5, 8, 8]$$

Non-symmetric Packings

Non-symmetric Packings



$[-23, 48, 49, 52]$.

Non-symmetric Packings

Non-symmetric Packings

Current best paramaterization of non-symmetric packings:

Non-symmetric Packings

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Theorem

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Given a general pair (x, y) with the criteria

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Non-symmetric Packings

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the form is

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Once again, not ideal. Not in terms of factorization.

Total Number of Packings

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The total packings of n is known:

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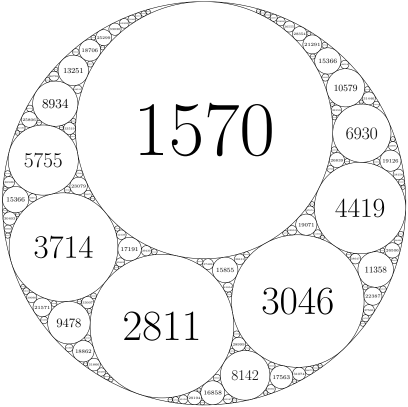
Thank You!



$[-1001, 1570, 2811, 3046]$

Images generated using James Rickard's Code.