Apollonian Circle Packings & the Local-Global Conjecture

Clyde Kertzer, with Summer Haag, Katherine E. Stange, and James Rickards

University of Colorado Boulder

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Descartes Quadruples

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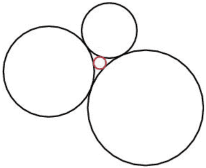

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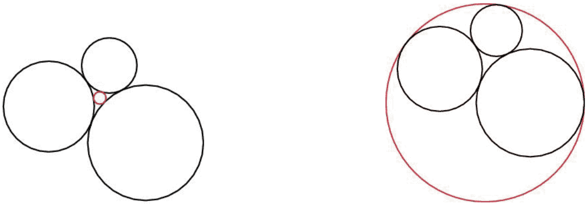

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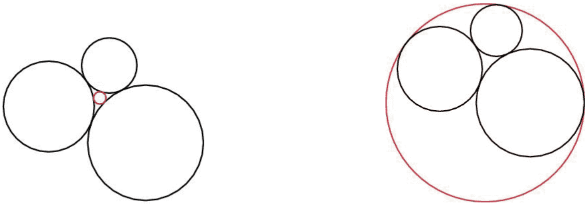

We can only have at most one "inverted" circle!

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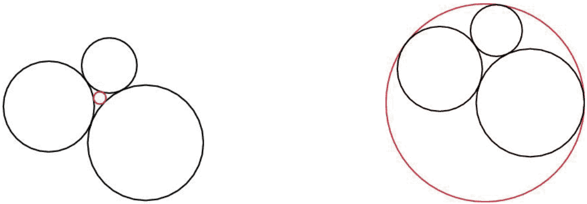

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Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

The Descartes Equation

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The *curvature* of a circle with radius r is defined to be $1/r$.

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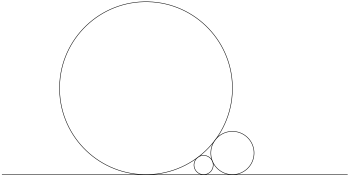

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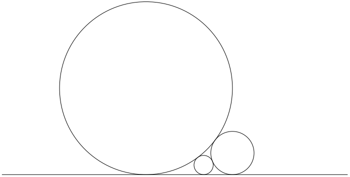

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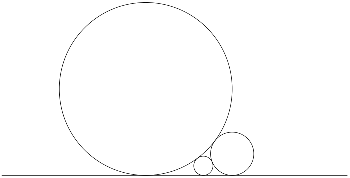

Circle with infinite radius

Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

Apollonian Circle Packings

Clyde Kertzer, with Summer Haag, Katherine E. Stange, and James Rickards

If a, b, c, d are integers, the rest are also integers!

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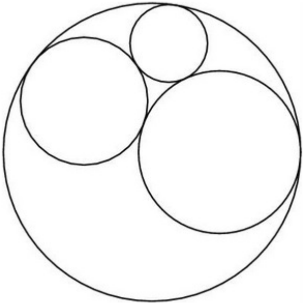

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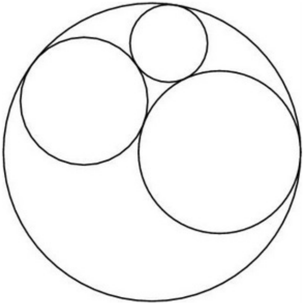

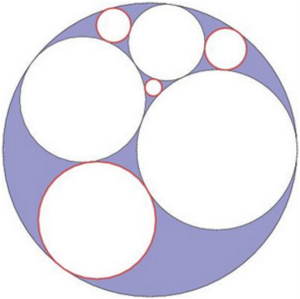

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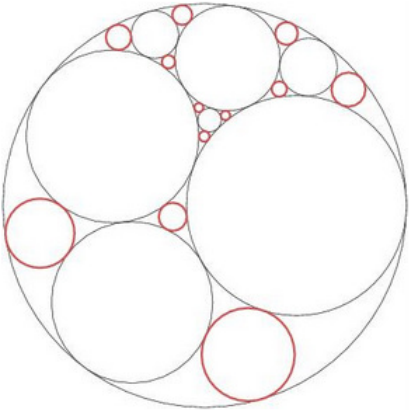

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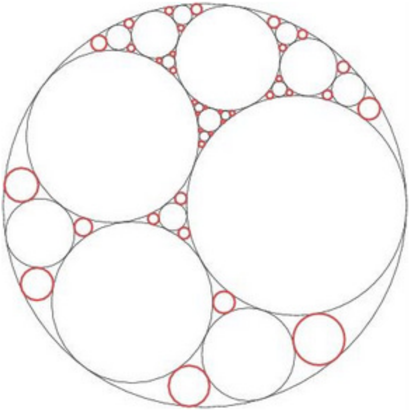

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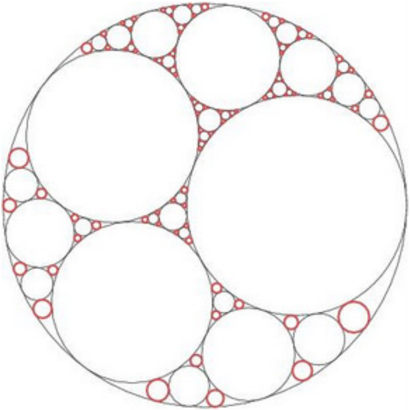

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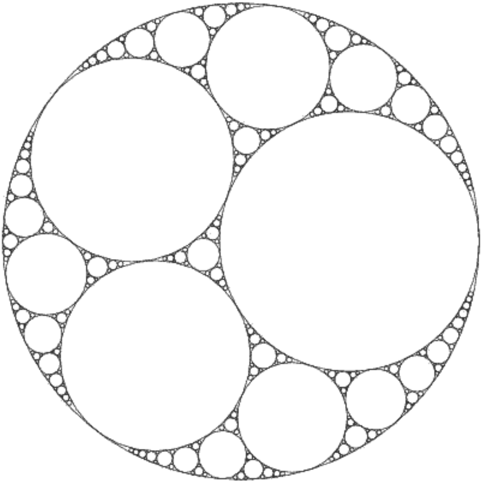

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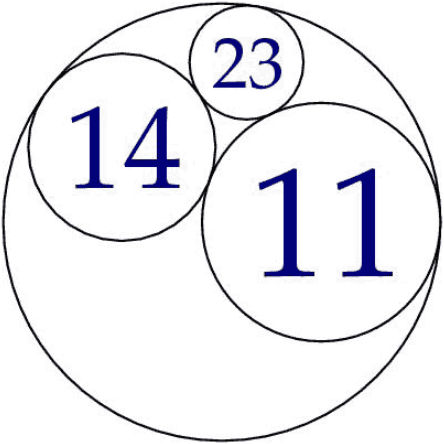

$[-6, 11, 14, 23]$[1]

[1] Images from: AMS "When Kissing Involves Trigonometry"

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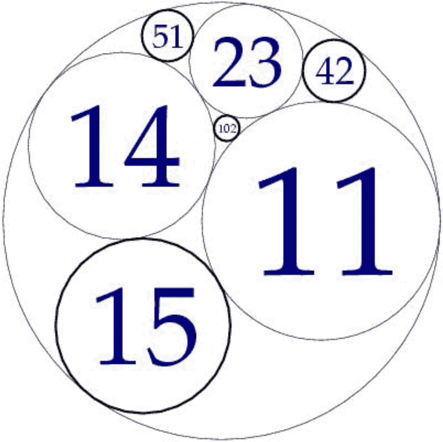

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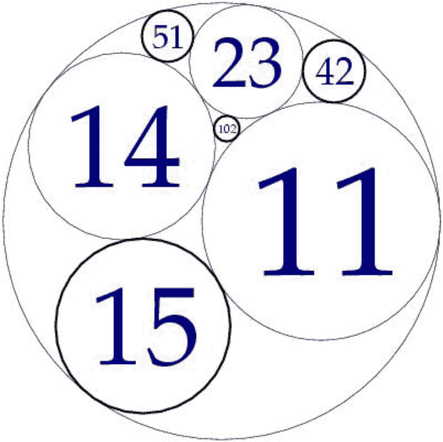
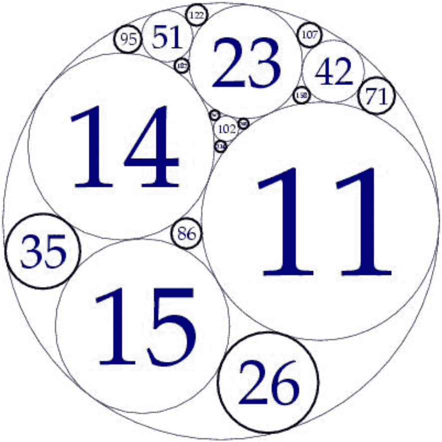


$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

Apollonian Circle Packings



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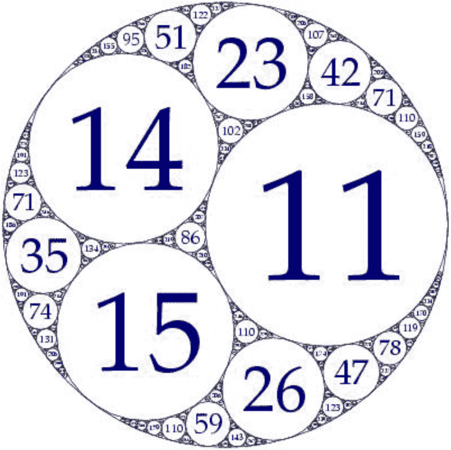

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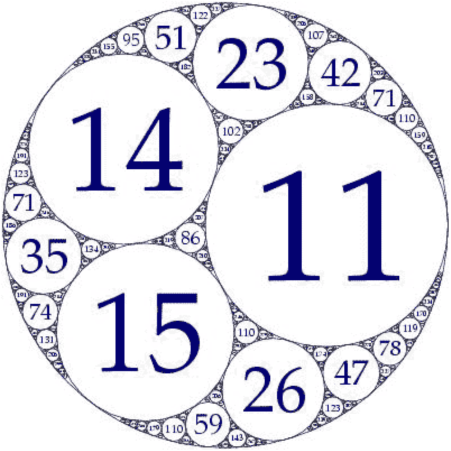

Notice: Once $-6, 11, 14, 15$ are set, no room for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, ...

Curvatures Mod 5

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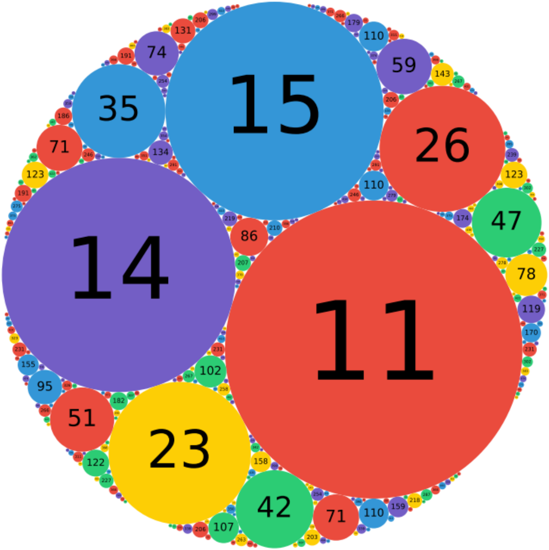

Curvatures Mod 3

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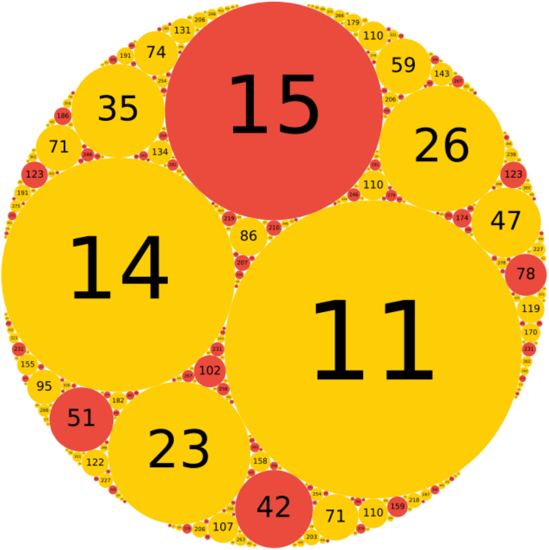

Allowed Residues

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Theorem (Fuchs)

If a congruence obstruction appears, then it appears modulo 24.

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Type	Allowed Residues
$(6, 1)$	$0, 1, 4, 9, 12, 16$
$(6, 5)$	$0, 5, 8, 12, 20, 21$
$(6, 13)$	$0, 4, 12, 13, 16, 21$
$(6, 17)$	$0, 8, 9, 12, 17, 20$
$(8, 7)$	$3, 6, 7, 10, 15, 18, 19, 22$
$(8, 11)$	$2, 3, 6, 11, 14, 15, 18, 23$

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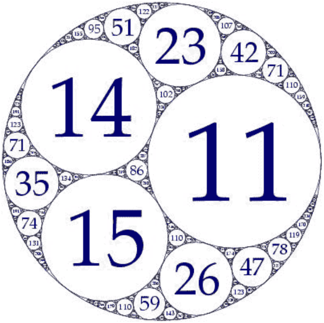

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Local-to-global

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Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden 2011)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

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The number of missing curvatures up to N is at most $O\left(N^{1-\eta}\right)$ for some effectively computable $\eta > 0$.

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Body of work by Graham-Lagarias-Mallows-Wilks-Yan, Sarnak, Bourgain-Fuchs, Bourgain-Kontorovich, Fuchs-S.-Zhang

Theoretical Tool: Quadratic Forms

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where f_a is a primitive integral binary quadratic form of discriminant $-4a^2$ associated to the 'mother circle'.

Computational Evidence

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Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } (-1, 2, 2, 3)$$
$$5 \cdot 10^8 \text{ for } (-11, 21, 24, 28)$$

and observed for $(-11, 21, 24, 28)$, there were still a small number (up to 0.013%) of missing curvatures in the range $\left(4 \cdot 10^8, 5 \cdot 10^8\right)$ for residue classes $0, 4, 12, 16 \bmod 24$.

Summer 2023 REU

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1. Fix a pair of curvatures, and study what packings contain them.

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3. Local-global: finitely many black dots on any row or column.

Typical graph

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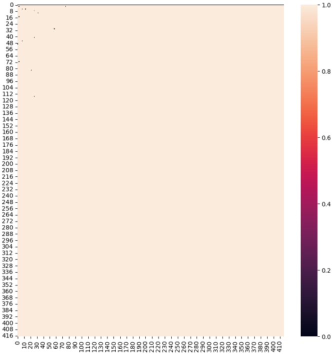

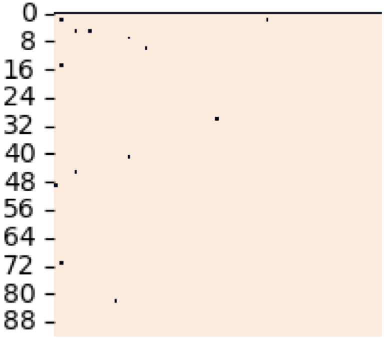

Residue classes 0 (mod 24) and 12 (mod 24) (Summer Haag)

One weird graph

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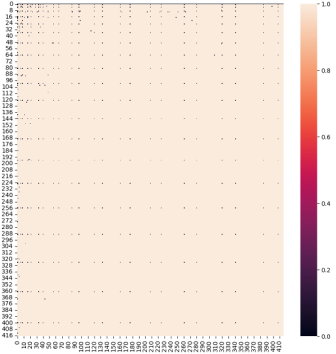

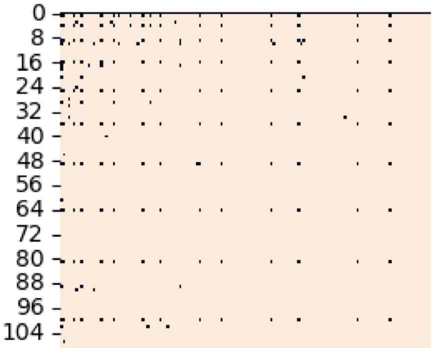

Residue classes 0 (mod 24) and 8 (mod 24) (Summer Haag)

The conjecture is false

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Theorem (Haag-Kertzer-Rickards-S.)

The Apollonian circle packing generated by quadruple $(-3, 5, 8, 8)$ *has no square curvatures.*

A circle has a 'residuosity'

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4. Define $\chi_2(\mathcal{C}) = 1$ if residues, -1 otherwise.

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4. So $\chi_2(\mathcal{C})$ is independent of the choice of circle $\mathcal{C}$.

There are no squares in the packing

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1. In base quadruple $(-3, 5, 8, 8)$, compute

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2. So no circle can be tangent to a square.

New invariants of a packing

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satisfies $\chi_4(\mathcal{C})^2 = \chi_2(\mathcal{C})$,
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The values of χ_2 and χ_4 determine the quadratic and quartic obstructions respectively.

The New Conjecture

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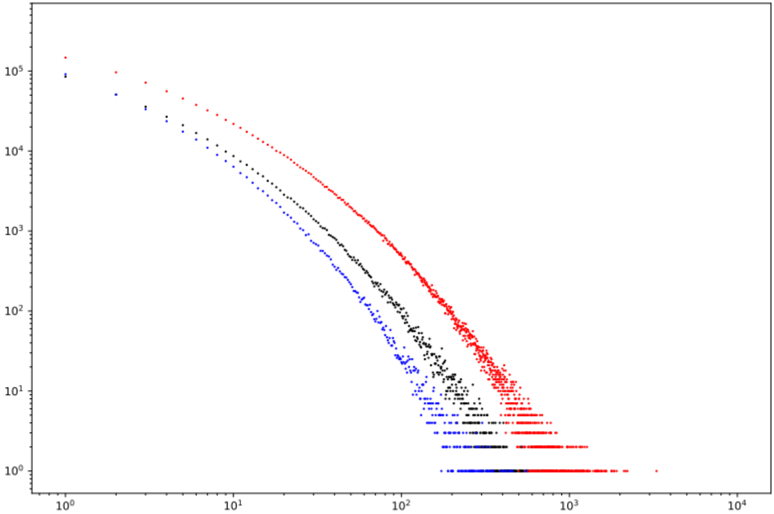
The type of a packing implies the existence of certain quadratic and quartic obstructions:

Type	n^2 Obstructions	n^4 Obstructions	L-G false	L-G open
$(6,1,1,-1)$		$n^4, 4n^4, 9n^4, 36n^4$	$0, 1, 4, 9, 12, 16$	
$(6,1,-1)$	$n^2, 2n^2, 3n^2, 6n^2$		$0, 1, 4, 9, 12, 16$	
$(6,5,1)$	$2n^2, 3n^2$		$0, 8, 12$	$5, 20, 21$
$(6,5,-1)$	$n^2, 6n^2$		$0, 12$	$5, 8, 20, 21$
$(6,13,1)$	$2n^2, 6n^2$		0	$4, 12, 13, 16, 21$
$(6,13,-1)$	$n^2, 3n^2$		$0, 4, 12, 16$	$13, 21$
$(6,17,1,1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0, 9, 12$	$8, 17, 20$
$(6,17,1,-1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0, 9, 12$	$8, 17, 20$
$(6,17,-1)$	$n^2, 2n^2$		$0, 8, 9, 12$	$17, 20$
$(8,7,1)$	$3n^2, 6n^2$		$3, 6$	$7, 10, 15, 18, 19, 22$
$(8,7,-1)$	$2n^2$		18	$3, 6, 7, 10, 15, 19, 22$
$(8,11,-1)$	$2n^2, 3n^2, 6n^2$		$2, 3, 6, 18$	$11, 14, 15, 23$

Sporadic curvatures dropping off



Thank You!

All images generated using James Rickard's Code.