Overview of Local-to-Global Conjecture for Apollonain Circle Packings

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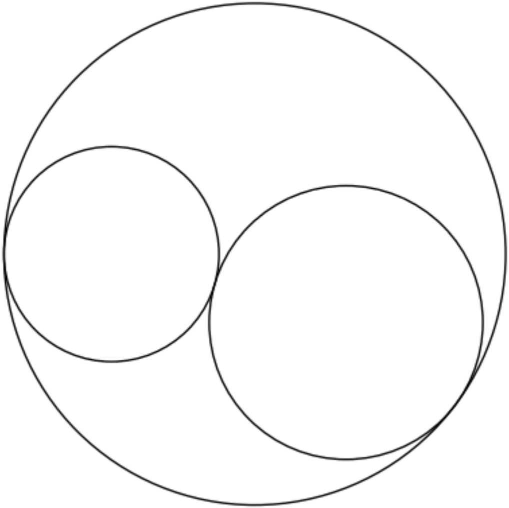
Apollonian Circle Packings



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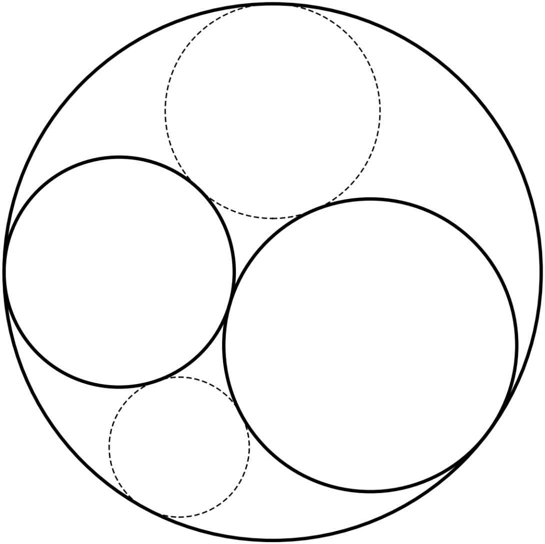

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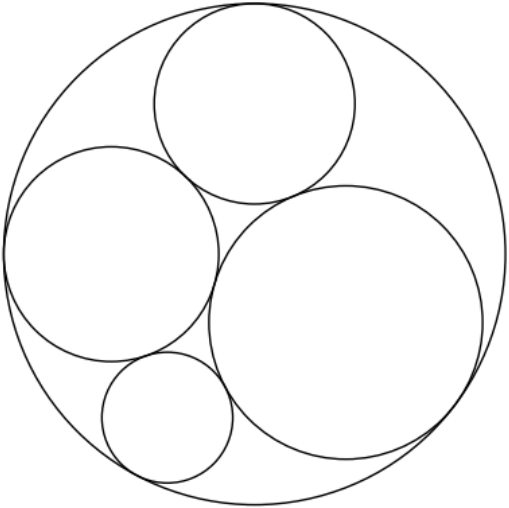

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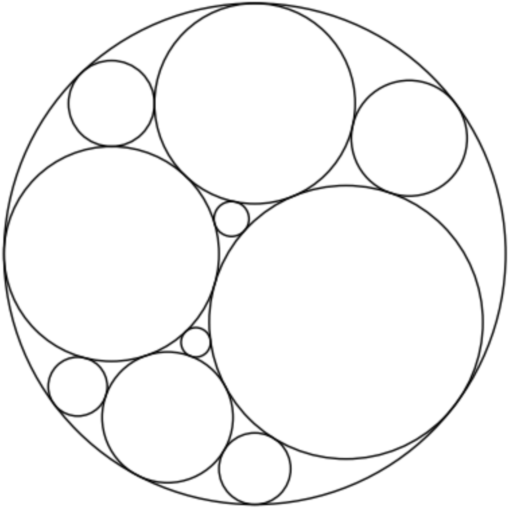

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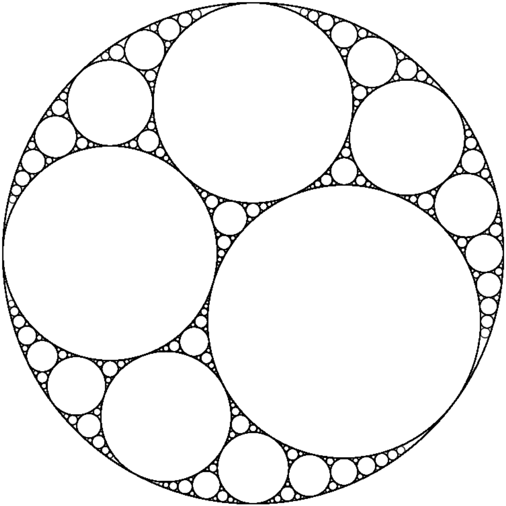

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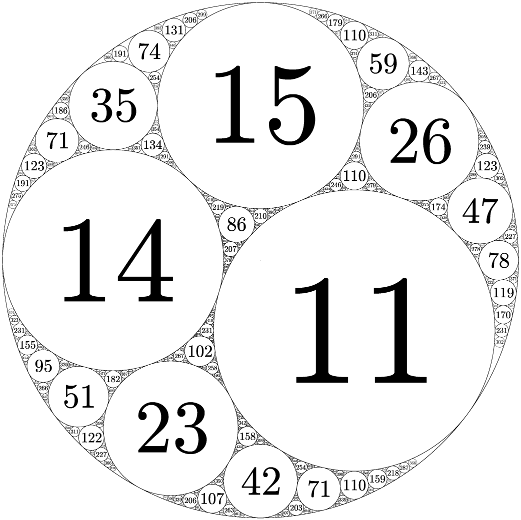

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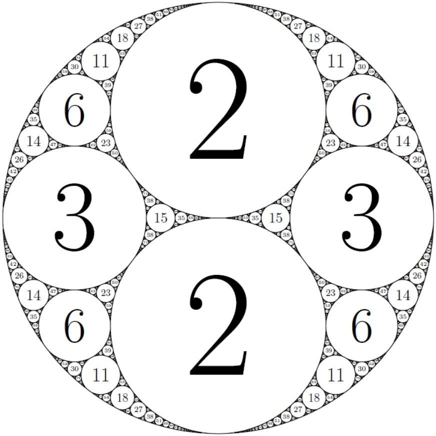
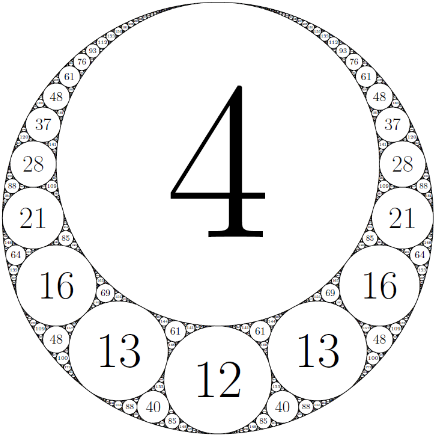


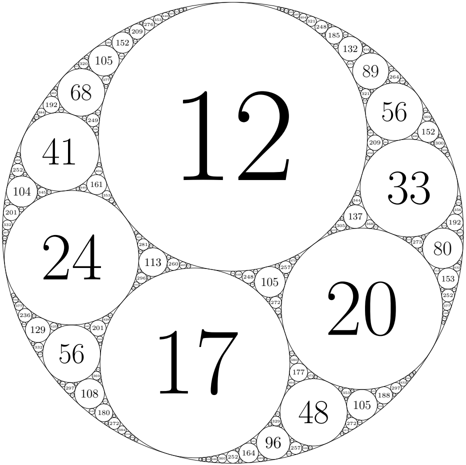
$[-1, 2, 2, 3]$

Apollonian Circle Packings



$[-3, 4, 12, 13]$

Apollonian Circle Packings



$[-7, 12, 17, 20]$

Curvature

The *curvature* of a circle with radius r is $1/r$.

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Negative curvature means the interior and exterior are flipped

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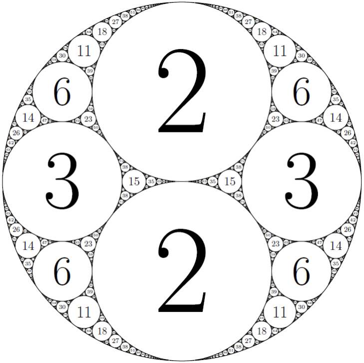
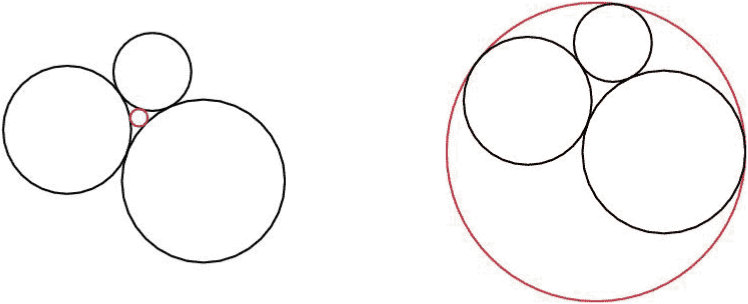
$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$



$[-1, 2, 2, 3]$

Theorem of Elizabeth and Descartes



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Given a, b, c, there exists d, d' with

$$d = 2(a + b + c) - d'$$

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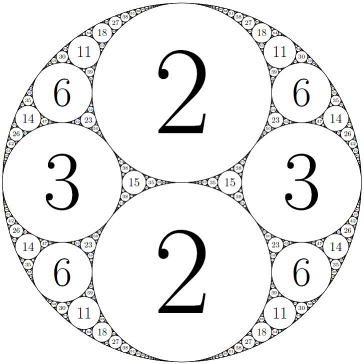
Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d′ have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Moreover, $d + d' = 2(a+b+c)$.

So if $a, b, c, d \in \mathbb{Z}$ then all curvatures $\in \mathbb{Z}$



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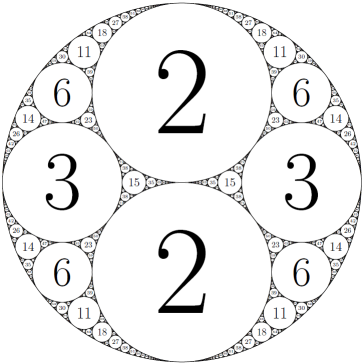

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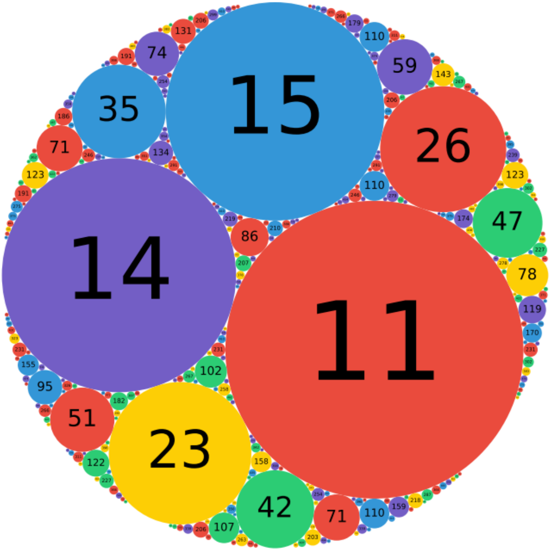
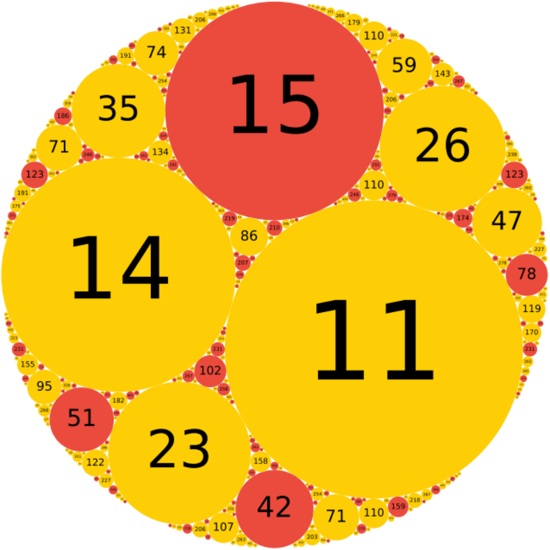
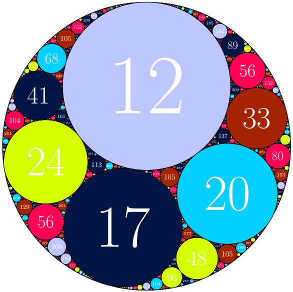
Curvatures Mod 5



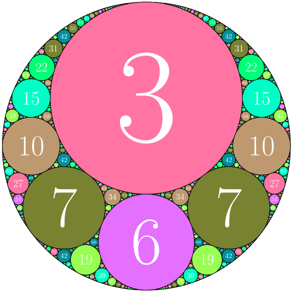
Curvatures Mod 3



Admissible Residues



$[-7, 12, 17, 20]$ colored mod 24



$[-2, 3, 6, 7]$ colored mod 24

Admissible Residues

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residues mod 24
0,1,4,9,12,16
0,5,8,12,20,21
0,4,12,13,16,21
0,8,9,12,17,20
3,6,7,10,15,18,19,22
2,3,6,11,14,15,18,23

Admissible Residues

Type	residues mod 24
a	0,1,4,9,12,16
b	0,5,8,12,20,21
c	0,4,12,13,16,21
d	0,8,9,12,17,20
e	3,6,7,10,15,18,19,22
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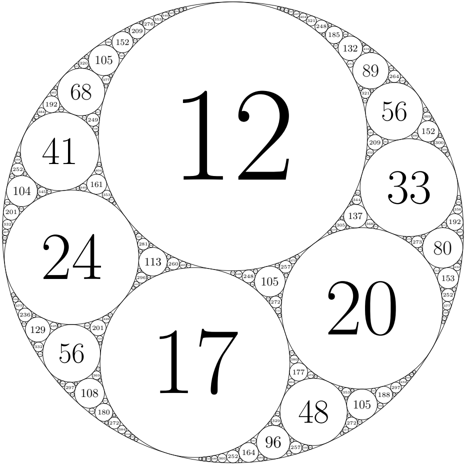
Theorem (Fuchs, 2010)

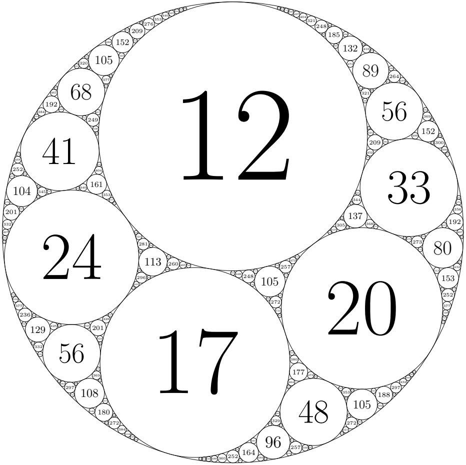
If a congruence obstruction appears, then it appears mod 24.

Missing curvatures



$[-7, 12, 17, 20]$ type d

Missing curvatures



$[-7, 12, 17, 20]$ type d $\implies$ $0, 8, 9, 12, 17, 20$

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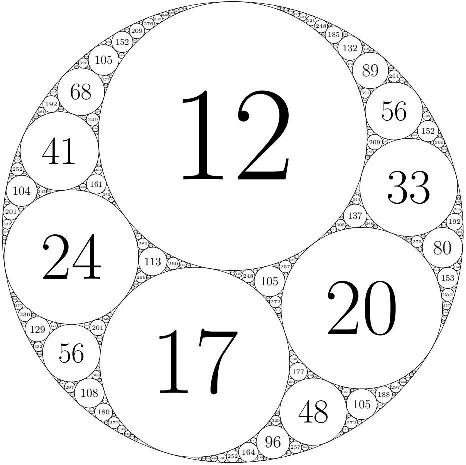

$[-7, 12, 17, 20]$ type d $\implies$ $0, 8, 9, 12, 17, 20$: no room for 8,9,32,...

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Conjecture (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden 2011):

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Conjecture (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden 2011):

Eventually, all admissible curvatures appear in a primitive Apollonian circle packing

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For $[-11, 21, 24, 28]$, there were still a small number (up to 0.013%) of missing curvatures in the range $(4 \cdot 10^8, 5 \cdot 10^8)$ for residue classes $0, 4, 12, 16 \pmod{24}$

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  - Since 0 and 12 can appear in the same packing compare integers in those residue classes
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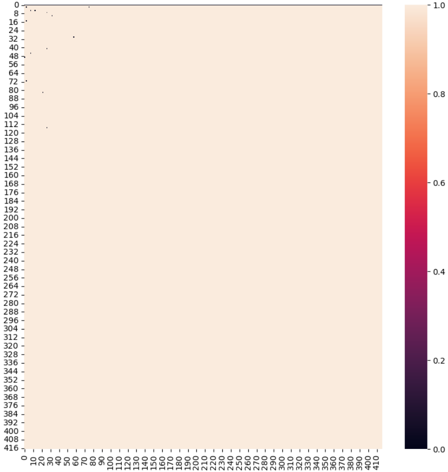
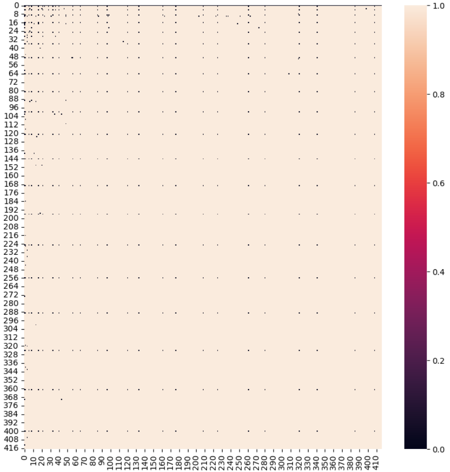
- For an acceptable pair, black dot if no packing contains them
- Local-to-global: finitely many black dots for a row or column

Usual Graph



Residue classes: 0 (mod 24) and 12 (mod 24)

Weird Graph



Residue classes: 0 (mod 24) and 8 (mod 24)

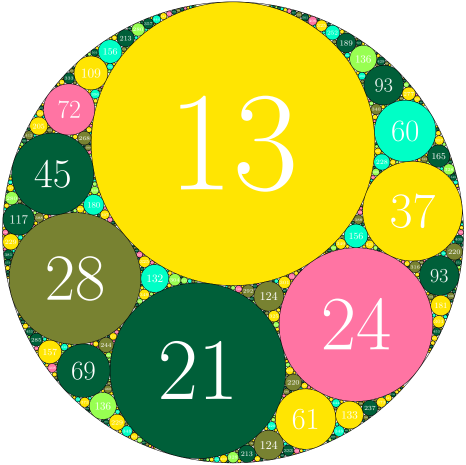
Local-to-global conjecture is false

Theorem (Haag-Kertzer-Stange-Rickards) Curvatures of the form $12k^2$ and $8n^2$ with $n \equiv 1, 2 \pmod 3$ will never appear in the same packing

The New Conjecture

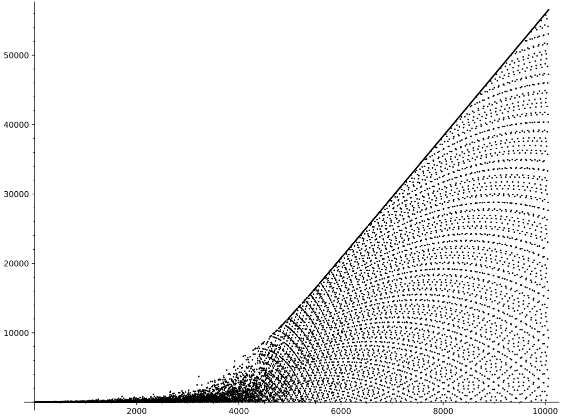
Type	Quadratic	Quartic	L-G false	L-G open
a				0, 1, 4, 9, 12, 16
a		$n^4, 4n^4, 9n^4, 36n^4$	0, 1, 4, 9, 12, 16	
a	$n^2, 2n^2, 3n^2, 6n^2$		0, 1, 4, 9, 12, 16	
b	$2n^2, 3n^2$		0, 8, 12	5, 20, 21
b	$n^2, 6n^2$		0, 12	5, 8, 20, 21
c	$2n^2, 6n^2$		0	4, 12, 13, 16, 21
c	$n^2, 3n^2$		0, 4, 12, 16	13, 21
d	$3n^2, 6n^2$	$9n^4, 36n^4$	0, 9, 12	8, 17, 20
d	$3n^2, 6n^2$	$n^4, 4n^4$	0, 9, 12	8, 17, 20
d	$n^2, 2n^2$		0, 8, 9, 12	17, 20
e	$3n^2, 6n^2$		3, 6	7, 10, 15, 18, 19, 22
e	$2n^2$		18	3, 6, 7, 10, 15, 19, 22
f				2, 3, 6, 11, 14, 15, 17, 23
f	$2n^2, 3n^2, 6n^2$		2, 3, 6, 18	11, 14, 15, 23

Apollonian Circle Packings



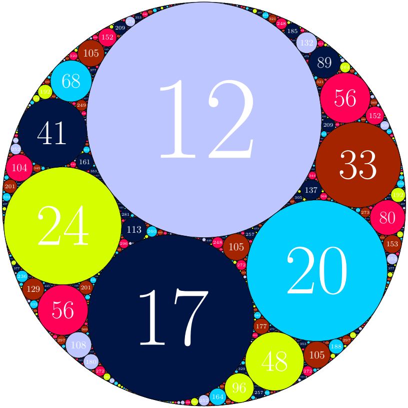
$[-8, 13, 21, 24]$

Some evidence



Successive differences of missing curvatures in the packing $[-4, 5, 20, 21]$. The quadratic families $2n^2$ and $3n^2$ begin to predominate. (There are 3659 numbers $< 10^{10}$, and occur increasingly sparsely.)

Thank You!!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The proof of $d + d' = a + b + c$

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Writing the Descartes equation as a quadratic polynomial in d we get

$$2\left(a^2 + b^2 + c^2 + d^2\right) - (a + b + c + d)^2 = 0$$
$$d^2 - 2d(a + b + c) + \left(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac\right) = 0.$$

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The quadratic formula gives

$$d=(a+b+c)\pm\frac{\sqrt{4(a+b+c)^2-4\left(a^2+b^2+c^2-2ab-2bc-2ac\right)}}{2}$$
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The quadratic formula gives

$$d = (a + b + c) \pm \frac{\sqrt{4(a + b + c)^2 - 4\left(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac\right)}}{2}$$
$$= a + b + c \pm 2\sqrt{ab + bc + ca}.$$

Thus, there are two options for d. Their sum is $2(a + b + c)$. □